Stochastic Analysis for the Complex Monge-Ampère Equation. (An Introduction to Krylov's Approach.)

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STOCHASTIC ANALYSIS FOR THE COMPLEX MONGE-AMPÈRE EQUATION

(AN INTRODUCTION TO KRYLOV'S APPROACH)

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We here gather in a single note several original probabilistic works devoted to the analysis of the $\mathcal{C}^{1,1}$ regularity of the solution to the possibly degenerate complex Monge-Ampère equation. The whole analysis relies on a probabilistic writing of the solution as the value function of a stochastic optimal control problem. Such a representation has been introduced by Gaveau [3] in the late 70's and used in an exhaustive way by Krylov in a series of papers published in the late 80's up to the final paper [7] in which the $\mathcal{C}^{1,1}$-estimate is eventually established. All the arguments we here use follow from these seminal works.

Nota Bene. This is an expanded version of the notes I prepared for a series of lectures I delivered in LATP, Marseille, in december 2009.

1. Introduction

Background. This Chapter is devoted to the stochastic analysis of the possibly degenerate Monge-Ampère equation and specifically to the probabilistic proof of the $\mathcal{C}^{1,1}$-estimate of the solutions under some suitable assumption.

For a complete review of the stakes of such a result, we refer the reader to Chapters 0 and 1 by V. Guedj and A. Zeriahi: we here focus on the probabilistic counterpart only and keep silent about the geometric motivations that are hidden behind.

The idea of understanding the complex Monge-Ampère equation from a probabilistic point of view goes back to the earlier paper by Gaveau [3] in the late 70's. Therein, the solution is shown to write as the value function of a stochastic optimal control problem, i.e. as the minimal value of some averaged cost computed along the trajectories of different diffusion processes evolving inside the underlying domain.

In some sense, this representation formula is a *compact* (or *closed*) representation formula that appears as a generalization of the Kolmogorov formula for the heat equation: the solution of the heat equation may be expressed as some averaged value computed along the trajectories of the Brownian motion. Brownian motion might be understood as follows: at any given time and at any given position, the diffusive particle at hand moves at random, independently of the past and in an isotropic way. Actually, Kolmogorov formula extends to linear (say to simplify purely) second-order partial differential equations with a variable diffusion coefficient: the solution is then understood as some averaged value computed along the solution of a differential equation of stochastic type driven by the coefficient of the PDE at hand. This appears as a stochastic method of characteristics: at any given time and at any given position, the diffusive particle associated with the stochastic differential equation moves at random, independently of the past, but in a non-isotropic way; the most likely directions are given by the main eigenvectors of the diffusion matrix at the current point. In the case of Monge-Ampère, the story might read as follows: at any given time and at any given position, the particle at hand moves at random, independently of the past, and the diffusion coefficient is chosen among all the possible diffusion coefficient of determinant 1 according to some local optimization criterion or, equivalently, to some local cost.

Purpose of the Note. In his paper, Gaveau managed to derive some Hölder continuity property of the solution to Monge-Ampère from the probabilistic formulation, but the exhaustive use of the formula for the analysis of the regularity of the derivatives goes back to Krylov. The reference paper on the subject is [7]: the solution is shown to be $\mathcal{C}^{1,1}$ on the whole domain (i.e. up to the boundary) under some suitable assumption that may include the degenerate case. Basically, it applies to a much more general framework than the Monge-Ampère one: it applies to a general class of Hamilton-Jacobi-Bellman equations, i.e. to a general class of equations summarizing the dynamics of the value function of some stochastic optimization problem.

Actually, the paper [7] is not *self-contained.* It must be seen as the conclusion of a series of papers initiated in the 80's: see, among others, [5], [6], [8] and, finally, [7]. This note is an attempt to gather in a single manuscript most of the ingredients of the whole proof, at least in the specific case of Monge-Ampère: from the basic rules of stochastic calculus to the detailed computations of the final estimate of the first- and second-order derivatives.

However, the proof we here provide is a bit different from the original one and may appear as less straightforward. In some sense, the objective is here both mathematical and... pedagogical: the idea is both to provide an *almost* complete and self-contained proof of the $\mathcal{C}^{1,1}$ estimate and to explain to the reader the way we are following to reach it.

A Short Review of the Strategy. The arguments used by Krylov have been developed since the 70's. Some of them may be found in the seminal work by Malliavin [11] and [12], even if used differently. In short, Malliavin initiated a program to prove by means of stochastic arguments only the *Sum of Squares Theorem* by Hörmander: *Sum of Squares Theorem* provides some sufficient condition on the Lie algebra generated by the vector fields of a possibly degenerate diffusion matrix to let the corresponding operator be hypoelliptic. The program consists in an exhaustive analysis of the stochastic flow generated by the associated differential equation of stochastic type. (For the purely Laplace operator, the flow is trivial since the current diffusion process reduces to a Brownian motion plus a starting point.) A part of the problem is then to investigate the regularity of the flow.

In the current framework, the main idea of Krylov consists in reducing the analysis of the $\mathcal{C}^{1,1}$ regularity of the solution to Monge-Ampère to a long-run analysis of the derivatives of the flow of the diffusion processes behind. Roughly speaking, the point is to control the first- and second-order derivatives of the flow both in time and in the optimization parameter. At first sight, it turns out to be really challenging. By the way, it is in some sense: stated under this form, the objective may not be reachable. Here is the key-point of the proof: the required long-run estimate of the derivatives of order one and two of the flow may be relaxed according to the underlying second-order differential structure. As an example, the analysis may benefit from some uniform ellipticity (or non-degeneracy) property: when applied to a non-degenerate linear second-order partial differential equation instead of the Monge-Ampère equation, the original required long-run estimate of the derivatives of the flow can be relaxed to a much more less restrictive version (and in fact can almost be cancelled) thanks to the non-degeneracy assumption itself. (The argument is explained in the note.) In the case of Monge-Ampère, the equation may degenerate, but the analysis may benefit from the description of the boundary: if the domain is strictly pseudo-convex, the original required long-run estimate of the derivatives of the flow can be relaxed as well (but cannot be cancelled); that is, strict pseudo-convexity plays the role of a *weak non-degeneracy assumption*. Finally, the analysis may also benefit from the Hamilton-Jacobi-Bellman formulation, i.e. from the writing of the Monge-Ampère equation as an equation deriving from a stochastic optimization problem: the structure is indeed kept invariant under some transformations of the optimization parameters. As explained below, this may also help to reduce the long-run constraint on the derivatives of the flow.

As mentioned, the way the required long-run constraint on the derivatives of the flow is relaxed is detailed in the note. At least, we may here specify the keyword only: *perturbation*. Indeed, the strategy is common to the Malliavin point of view and consists of a well-chosen perturbation of the original probabilistic representation. This is a general *meta*-principle in stochastic analysis: from a probabilistic point of view, regularity properties are understood through the reaction of the stochastic system under consideration to an external perturbation.

Main Result. In the end, the result we here prove is the following:

Theorem 1.1. *Let* ($\mathbf{A}$) *stand for the assumption:*

- *$\mathcal{D}$ is a bounded domain of $\mathbb{C}^d$, $d \geq 1$, described by some $\mathcal{C}^4$ function ψ in the neighborhood of $\bar{\mathcal{D}}$, i.e.*
$$\mathcal{D} := \big\{ z \in \mathbb{C}^d : \psi(z) > 0 \big\}.$$
- *The function ψ is assumed to be plurisuperharmonic in the neighborhood of $\bar{\mathcal{D}}$, i.e.*
$$\forall a \in \mathcal{H}_d^+ : \mathrm{Trace}(a) = 1, \quad \forall z \in \bar{\mathcal{D}}, \ \mathrm{Trace}\big(a D^2_{z,\bar{z}} \psi(z)\big) < 0,$$
where $\mathcal{H}_d^+$ stands for the set of non-negative Hermitian matrices of size $d \times d$.
- *The function ψ is non-singular in the neighborhood of the boundary of $\mathcal{D}$, i.e.*
$$\exists \delta > 0, \ \forall z \in \partial \mathcal{D}, \quad |D_z \psi(z)| \geq \delta.$$
- *f and g are two functions of class $\mathcal{C}^2$ and $\mathcal{C}^4$ on $\bar{\mathcal{D}}$ with values in $\mathbb{R}_+$ and $\mathbb{R}$ respectively.*

Then, under Assumption ($\mathbf{A}$), *there exists a function u from $\bar{\mathcal{D}}$ to $\mathbb{R}$, of class $\mathcal{C}^{1,1}$ on the whole $\bar{\mathcal{D}}$ (i.e. with Lipschitz first-order derivatives on the closure of the domain $\mathcal{D}$), plurisubharmonic, i.e.*

$$\forall a \in \mathcal{H}_d^+ : \mathrm{Trace}(a) = 1, \quad \text{a.e. } z \in \mathcal{D}, \ \mathrm{Trace}\big(a D^2_{z,\bar{z}} u(z)\big) \geq 0,$$

and

$$\det{}^{1/d}\big(D^2_{z,\bar{z}} u(z)\big) = \frac{f(z)}{d} \ \text{a.e. } z \in \mathcal{D}, \quad u(z) = g(z), \ z \in \partial \mathcal{D}, \tag{1.1}$$

i.e. u satisfies the Monge-Ampère equation on $\mathcal{D}$ with f^d (up to some normalizing constant) as source term and g as boundary condition. (Compare with Chapter 0, Section 1, by V. Guedj.)

Pay attention that Theorem 1.1 does not recover Theorem 1.3.1 in Chapter 1 by V. Guedj and A. Zeriahi (that holds for the ball only) since the boundary condition therein is $\mathcal{C}^{1,1}$ only.

Organization of the Note. The note is organized as follows. In Section 2, we explain the basic optimization principle on which the whole proof relies. In Sections 3 and 4, we introduce the Kolmogorov representation of the Dirichlet problem with constant coefficients by means of the Brownian motion. We then give a short overview of the basic rules of stochastic calculus. In Section 5, we introduce the probabilistic representation of Monge-Ampère, as originally considered by Gaveau. The program for the analysis of the representation is explained in Section 6. Section 7 is a short presentation of the differentiability properties of the flow of a stochastic differential equation. In Section 8, we give a first sketch of the proof of the $\mathcal{C}^1$-regularity. As explained therein, it fails for the second-order derivatives. The right argument is given in Section 9.

Useful Notation. Below, the gradient of a function is understood as a row vector and for any pair of vectors (x, y) (of the same dimension d) with real or complex coordinates, the notation $\langle x, y\rangle$ stands for $\sum_{i=1}^d x_i y_i$.

2. Hamilton-Jacobi-Bellman Formulation

We here introduce the Hamilton-Jacobi-Bellman formulation of the Monge-Ampère equation.

2.1. **Optimization Problem.** Generally speaking, Hamilton-Jacobi-Bellman equations describe the dynamics – in space only for a stationary problem and in time as well for an evolution equation – of the value function of an optimal (possibly stochastic) control problem.

In the specific case of Monge-Ampère, the Hamilton-Jacobi-Bellman formulation follows from a simple Lemma taken from the original article by Gaveau [3]:

Lemma 2.1. *Given a non-negative Hermitian matrix H of size $d \times d$, the determinant of H is the solution of the minimization problem:*

$$\det{}^{1/d}(H) = \frac{1}{d} \inf\bigl\{\mathrm{Trace}[aH]\ ;\ a \in \mathcal{H}_d^+,\ \det(a) = 1\bigr\}.$$

Proof. Up to a diagonalization, we may assume H to be diagonal. Denoting by $(\lambda_1, \dots, \lambda_d)$ its (non-negative real) eigenvalues, we obtain for some $a \in \mathcal{H}_d^+$

$$\mathrm{Trace}[aH] = \sum_{i=1}^d a_{i,i}\lambda_i.$$

Noting that the elements $(a_{i,i})_{1\le i\le d}$ are non-negative, the standard inequality between the arithmetic and geometric means yields

$$\frac{1}{d}\mathrm{Trace}[aH] \ge \bigl(\prod_{i=1}^d a_{i,i}\lambda_i\bigr)^{1/d} = \det{}^{1/d}(H)\bigl(\prod_{i=1}^d a_{i,i}\bigr)^{1/d}.$$

Finally, Hadamard inequality says that $\mathrm{Trace}[aH] \geq d \det^{1/d}(H)$, that is

$$\inf\{\mathrm{Trace}[aH]; a \in \mathcal{H}_d^+, \det(a) = 1\} \geq d \det^{1/d}(H).$$

To prove the equality between both quantities, we choose $a_{i,i} = \lambda_i^{-1} \det^{1/d}(H)$ (and $a_{i,j}$ equal to zero for i and j different) when H is non-degenerate (so that the *infimum* then reads as a *minimum*). In the degenerate case, it is sufficient to choose $a_{i,i} = \varepsilon$ when $\lambda_i > 0$ and $a_{i,i} = N$ when $\lambda_i = 0$, with ε small and N large to be chosen so that the determinant be equal to 1 (again, $a_{i,j}$ is set equal to 0 for i and j different). $\square$

Lemma 2.1 suggests us to write, at least formally, Monge-Ampère Eq. (1.1) under the form:

$$\sup_{a \in \mathcal{H}_d^+,\ \det(a)=1} \left[-\mathrm{Trace}[a D^2_{z,\bar{z}} u](z)\right] + f(z) = 0, \quad z \in \mathcal{D}. \tag{2.1}$$

(With the same boundary condition.) This formulation makes the family of diffusion operators $(\mathrm{Trace}[a D^2_{z,\bar{z}} \cdot])_{a \in \mathcal{H}_d^+,\ \det(a)=1}$ appear.

Roughly speaking, an equation driven by an *infimum* (or a *supremum*) taken over a family of second-order operators is called a second-order *Hamilton-Jacobi-Bellman* equation.

2.2. **First-Order Case.** We first explain how minimization (or maximization) may affect a family of first-order partial differential equations. In such a case, the resulting equation is called a first-order *Hamilton-Jacobi-Bellman* equation. Consider to this end a very simple one-dimensional evolution problem:

$$D_t u(t,x) - \sup_{a \in \mathbb{R},\ |a|=1} \left[a D_x u\right](t,x) = 0, \quad (t,x) \in (0,+\infty) \times \mathbb{R}, \tag{2.2}$$

with a given regular boundary condition $u(0,\cdot) = u_0(\cdot)$. This is a non-linear equation with

$$D_t u(t,x) - \left|D_x u\right|(t,x) = 0, \quad (t,x) \in (0,+\infty) \times \mathbb{R},$$

as explicit form.

The purpose is here to understand how the method of characteristics may write for such an equation. When the *parameter* or *control* a is frozen, the equation

$$D_t u(t,x) - a D_x u(t,x) = 0, \quad (t,x) \in (0,+\infty) \times \mathbb{R}, \tag{2.3}$$

is a simple transport equation with $-a$ as constant velocity, whose solution is explicitly known:

$$u(t,x) = u_0(x + at), \quad (t,x) \in [0,+\infty) \times \mathbb{R}.$$

Said differently, the initial shape u_0 is translated at velocity $-a$: as an example, the value of u at time t and a point $-at$ is $u_0(0)$. Said differently, the mapping $t \geq 0 \mapsto u(t, x - at)$ is constant.

Here, the linear mapping $t \geq 0 \mapsto x + at$ is called a *backward characteristic* of the transport equation (2.3).

Go now back to the general case. We understand that the *supremum* in Eq. (2.2) favours the velocity fields of same sign as the local spatial variation of the solution. Said differently, the possible characteristics must now be sought among paths driven by positive or negative speed according to the values of the gradient of the solution of the PDE. We thus consider paths of the form

$$x_t = x_0 + \int_0^t a_s ds, \quad t \geq 0, \tag{2.4}$$

where $(a_t)_{t\geq 0}$ is a (measurable) function with values in $\{-1, 1\}$ and x_0 is an arbitrary initial condition. The whole point is then to understand the behavior of the solution to the PDE along all these trajectories. To do so, wa may differentiate, at least formally, u along some $(x_t)_{t\geq 0}$ as in (2.4). For a given time $T > 0$ and some $t \in [0, T]$, we write

$$\begin{aligned}\frac{d}{dt}\big[u(T-t, x_t)\big] &= -D_t u(T-t, x_t) + a_t D_x u(T-t, x_t) \\ &= -|D_x u|(T-t, x_t) + a_t D_x u(T-t, x_t) \leq 0,\end{aligned}$$

by taking into account the equality $|a_t| = 1$. Therefore,

$$u(T, x_0) \geq u_0\Big(x_0 + \int_0^T a_s ds\Big),$$

that is

$$u(T, x_0) \geq \sup_{(a_t)_{0\leq t\leq T}:|a_t|=1} \Big[u_0\Big(x_0 + \int_0^T a_s ds\Big)\Big]. \tag{2.5}$$

Now, the formal choice $(a_t = \text{sign}[D_x u(T-t, x_t)])_{t\geq 0}$ says that equality might hold. We thus derive as a (possible) *closed* representation formula of u:

$$u(T, x_0) = \sup_{(a_t)_{0\leq t\leq T}:|a_t|=1} \big[u_0(x_T^a)\big], \tag{2.6}$$

with

$$x_t^a = x_0 + \int_0^t a_s ds, \quad t \geq 0.$$

The argument is here formal only. However, it suggests some possible *closed* representation for the solution of Eq. (2.2) as the value function of a deterministic control problem: the so-called *control parameter* is of the form $(a_t)_{t\geq 0}$ with $|a_t| = 1$, $t \geq 0$, and the resulting controlled path is of the form $(x_t^a)_{t\leq 0}$. We stress out that the *supremum* in (2.2) is kept preserved in the representation formula (2.6). This follows from a maximum principle argument: by the maximum principle, the solution to (2.2) is above the solution to any linear transport PDE with the same initial condition u_0 and with a (possibly time-dependent) velocity field of norm 1. (See (2.5).)

We also emphasize that the theory of viscosity solutions provides a rigorous framework to the formal argument we have here given. (See for example Chapter 2, Lemma 2.1, in the monograph by Barles [1].)

2.3. **Second-Order Equations.** Go now back to the Hamilton-Jacobi-Bellman formulation (2.1). In comparison with the previous subsection, we may distinguish two main differences. On the hand, Eq. (2.1) has a source term. On the other hand, the underlying operator is of second-order. (The reader may also notice that the equation is also stationary and that it is set on a bounded domain of the space only. We will come back to these two points later.)

Plugging a source term (say f in the right-hand side) in the Hamilton-Jacobi formulation (2.2) would not really modify the analysis we just performed. In a such a case, the right form of (2.6) would be

$$u(T, x_0) = \sup_{(a_t)_{0 \le t \le T} : |a_t| = 1} \left[u_0(x_T^a) + \int_0^T f(x_t^a) dt \right]. \tag{2.7}$$

(That is, the source term would be integrated along the controlled trajectories.)

Replacing the first-order operator by a second-order one is actually much more difficult to understand. To do so, the first point consists in going back to the frozen problem without any optimization, i.e. to the case when the diffusion coefficient in (2.1) is given by some fixed $a \in \mathcal{H}_d^+$, and then in seeking for the right characteristics in that framework.

Under this form, the problem is not well-posed. The whole point is the following: for a second-order operator, there are no *true characteristics*; the only possible way to obtain a closed formula for the solution consists in introducing an additional parameter, i.e. some randomness, and then in considering *random characteristics*. This follows from some scale factors: there is no way to balance, in a single differentiation, first-order terms in time and in space and second-order terms in space. More precisely, to balance first-order terms in time and second-order terms in space, the point is to introduce some characteristics with unbounded variation and, in fact, characteristics that are not absolutely continuous w.r.t. the Lebesgue measure. Randomness may be useless for the construction of such trajectories: as we will see below, randomness permits to get rid of some parasitic terms of order one by a simple integration w.r.t. to the underlying probability measure.

The typical case is the purely Laplace one. When a matches the identity matrix I_d, the operator $\mathrm{Trace}[D^2_{z,\bar{z}} \cdot]$ admits the complex Brownian motion of dimension d as *random characteristic*. Actually, $\mathrm{Trace}[D^2_{z,\bar{z}} \cdot]$ may be expanded in real coordinates as

$$\mathrm{Trace}[D^2_{z,\bar{z}} \cdot] = \frac{1}{4} \left[\Delta_{x,x} + \Delta_{y,y} \right],$$

so that it is equivalent to consider the real Brownian motion of dimension $2d$ as *random characteristic*: Brownian motion is the right *stochastic process* associated with the heat equation.

3. Brownian Motion

We first explain what Brownian motion is in the simplest case when the dimension is 1.

3.1. **Gaussian Density.**

The connection between Brownian motion and heat equation is well-understood through the so-called marginal laws, that is the laws of the positions of a Brownian motion at a given time. Recall indeed that the time–space heat equation in dimension 1

$$D_t u(t,x) - \frac{1}{2} D^2_{x,x} u(t,x) = 0, \quad (t,x) \in (0,+\infty) \times \mathbb{R}, \tag{3.1}$$

with an initial condition of the form $u(0,\cdot) = u_0(\cdot)$ admits as solution (say if u_0 is bounded and continuous)

$$u(t,x) = \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} u_0(x-y) \exp\left(-\frac{|y|^2}{2t}\right) dy, \quad (t,x) \in (0,+\infty) \times \mathbb{R}. \tag{3.2}$$

Said differently, the solution may be expressed as the convolution of the initial condition by the Gaussian density of zero mean and of variance t, i.e. the function

$$y \in \mathbb{R} \mapsto \frac{1}{\sqrt{2\pi t}} \exp\big(-\frac{|y|^2}{2t}\big) dy.$$

The density is here said to be of zero mean and of variance t since

$$\frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} y \exp\big(-\frac{|y|^2}{2t}\big) dy = 0$$
$$\frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} y^2 \exp\big(-\frac{|y|^2}{2t}\big) dy = t.$$

(The second result follows from a simple change of variable .)

Convolution by a Gaussian kernel may be expressed in a simple probabilistic way. Indeed, if $(\Omega, \mathcal{F}, \mathbb{P})$ denotes a complete[1] probability space and $(B_t)_{t\geq 0}$ a family of random variables (i.e. of measurable functions from $(\Omega, \mathcal{F})$ to $\mathbb{R}$ endowed with its Borel sets) such that, for any $t > 0$, B_t has a Gaussian density of zero mean and variance t, i.e. (below, $\mathbb{E}$ stands for the expectation)

$$\begin{aligned} \forall f \in \mathcal{C}_b(\mathbb{R}), \quad \mathbb{E}\big[f(B_t)\big] &= \int_{\Omega} f(X_t(\omega)) d\mathbb{P}(\omega) \\ &= \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} f(y) \exp\big(-\frac{|y|^2}{2t}\big) dy, \end{aligned}$$

and $\mathbb{P}\{B_0 = 0\} = 1$, then

$$u(t,x) = \mathbb{E}\big[u_0(x + B_t)\big], \quad t \geq 0. \tag{3.3}$$

3.2. **Dynamics.** The connection we just gave between heat equation and Gaussian variables is actually too much “static” to be fully relevant. Nothing is said about the joint behavior of the variables $(B_t)_{t\geq 0}$ ones with others.

To understand the dynamics, we use a discretization *artifact*. Assume indeed that we are applying a finite difference numerical scheme to solve heat equation (3.1). Specifically, for a small time step Δt and a small spatial step Δx, assume that we are seeking for a family of reals $(u_{n,k})_{n\in\mathbb{N},k\in\mathbb{Z}}$ approximating the “true” values $(u(n\Delta t, k\Delta x))_{k\in\mathbb{Z}}$. A common scheme consists in defining $(u_{n,k})_{n\in\mathbb{N},k\in\mathbb{Z}}$ through the iterative procedure

$$\frac{u_{n+1,k} - u_{n,k}}{\Delta t} = \frac{1}{2} \frac{u_{n,k+1} + u_{n,k-1} - 2u_{n,k}}{\Delta x^2}, \quad n \in \mathbb{N},\ k \in \mathbb{Z}, \tag{3.4}$$

with $u_{n,k} = u_0(k\Delta x)$ as initial condition. Obviously, in the above equation, the left-hand side is understood as an approximation of the time-derivative of u and the right-hand side of its second-order spatial derivative.

We can write (3.4) as

$$u_{n+1,k} = \big(1 - \frac{\Delta t}{\Delta x^2}\big) u_{n,k} + \frac{\Delta t}{\Delta x^2} \frac{u_{n,k+1} + u_{n,k-1}}{2}.$$

[1]The completeness is used in the sequel.

Choosing $\Delta t = \Delta x^2$, we obtain the simpler formula

$$u_{n+1,k} = \frac{u_{n,k+1} + u_{n,k-1}}{2}, \quad n \in \mathbb{N},\ k \in \mathbb{Z} \tag{3.5}$$

Replace now the approximating values $(u_{n,k})_{k\in\mathbb{Z},n\geq 0}$ in (3.5) by the true quantities and write

$$\begin{aligned} u\big((n+1)\Delta t, k\Delta x\big) &\approx \frac{u\big(n\Delta t, (k+1)\Delta x\big) + u\big(n\Delta t, (k-1)\Delta x\big)}{2} \\ &= \mathbb{E}\big[u(n\Delta t, k\Delta x + \Delta x\,\varepsilon)\big], \end{aligned}$$

where ε is a random variable taking the values 1 and -1 with probability $1/2$. Notice that it is possible to repeat the argument by approximating $u(n\Delta t, \cdot)$ with a new expectation (computed w.r.t. a new random variable, independent of ε). Therefore,

$$u\big((n+1)\Delta t, k\Delta x\big) \approx \mathbb{E}\big[u\big((n-1)\Delta t, k\Delta x + \Delta x(\varepsilon_1 + \varepsilon_2)\big)\big],$$

where ε_1 and ε_2 are two independent random variables taking the values 1 and -1 with probability $1/2$. Iterating the procedure N times, we deduce that

$$u\big(N\Delta t, k\Delta x\big) \approx \mathbb{E}\big[u\big(0, k\Delta x + \Delta x(\varepsilon_1 + \varepsilon_2 + \cdot + \varepsilon_N)\big)\big]. \tag{3.6}$$

Clearly, the symbol $\approx$ is not really meaningful because of the numerous approximations we just performed. However, choosing to simplify $k = 0$ and $N\Delta t = 1$, so that $\Delta x = N^{-1/2}$ since $\Delta t = \Delta x^2$, we understand that the random variable in the right-hand side in (3.6) has the form

$$N^{-1/2}\big[\varepsilon_1 + \varepsilon_2 + \cdot + \varepsilon_N\big].$$

Central Limit Theorem says that it converges, in the weak sense, towards the Gaussian law of zero mean and variance 1. (Here, weak convergence means weak convergence of probability measures.) In particular, passing to the limit in (3.6), we recover Eq. (3.3).

Actually, this non-rigorous argument says that the right structure for $(B_t)_{t\geq 0}$ in (3.3) is of independent increment type. Indeed, we understand that, on disjoint intervals, the underlying variables $(\varepsilon_n)_{n\geq 1}$ are asked to be independent. Moreover, the structure is stationary: randomness between times 0 and $t - s$ is the same in law as the randomness plugged into the system between times s and t. This says that the right choice for $(B_t)_{t\geq 0}$ is

Definition 3.1. *A family of random variables $(B_t)_{t\geq 0}$ is a Brownian motion starting from 0 if*

(1) $\mathbb{P}\{B_0 = 0\} = 1$,
(2) *for any $n \geq 1$, for any $t_0 = 0 < t_1 < t_2 < \cdots < t_n$, the increments B_{t_1}, $B_{t_2} - B_{t_1}$, $\cdots$, $B_{t_n} - B_{t_{n-1}}$ are independent,*
(3) *for any $0 < s < t$, the increment $B_t - B_s$ has a Gaussian law of zero mean and variance $t - s$.*
(4) *with probability 1, the paths $t \geq 0 \mapsto B_t(\omega)$ are continuous.*

The last condition is the most technical one: roughly speaking, it says that the differential structure associated with Brownian motion is local. Add also that, by definition, a Brownian motion starting from x is nothing else but x plus a Brownian motion starting from 0.

3.3. **Differential Rules.** To understand if Brownian motion is the right characteristic for heat equation, the point is to compute the infinitesimal variation of $(u(T-t,B_t))_{0\le t\le T}$, for a given $T>0$, where u is given by (3.1). We here expand by Taylor's formula

$$\begin{aligned}
&u(T-(t+h), B_{t+h}-B_t+B_t)\\
&= u(T-t,B_t) - D_t u(t,B_t)h + D_x u(t,B_t)(B_{t+h}-B_t)\\
&\quad + \frac{1}{2}D^2_{x,x}u(t,B_t)(B_{t+h}-B_t)^2 + \frac{1}{2}D^2_{t,t}u(t,B_t)h^2\\
&\quad - D^2_{t,x}u(t,B_t)(B_{t+h}-B_t)h + \dots
\end{aligned}$$

Expansion is given at least of order two: we aim to recover heat equation. (Moreover, it makes sense since u is regular away from the boundary.).

Actually, it is enough to stop the expansion at order two: by definition of a Brownian motion, $\mathbb{E}[(B_{t+h}-B_t)^2]=h$; using a simple Gaussian argument, this result may be generalized as $\mathbb{E}[(B_{t+h}-B_t)^{2p}]=C_p h^p$ for any integer p, the constant C_p being universal. In particular, the only term of order 1 in h among the derivatives of order two is the term in spatial derivatives. The others are of order $h^{3/2}$ and h^2. Therefore, we write

$$\begin{aligned}
&u(T-(t+h), B_{t+h}-B_t+B_t)\\
&= u(T-t,B_t) - D_t u(t,B_t)h + D_x u(t,B_t)(B_{t+h}-B_t)\\
&\quad + \frac{1}{2}D^2_{x,x}u(t,B_t)(B_{t+h}-B_t)^2 + \dots
\end{aligned} \tag{3.7}$$

Here, we wish to replace $(B_{t+h}-B_t)^2$ by h. Using a Gaussian argument again,

$$\mathbb{E}\Big[\big((B_{t+h}-B_t)^2-h\big)^2\Big] = 2h^2.$$

Clearly, this does not show that the term $(B_{t+h}-B_t)^2-h$ is less than h. However, on the long run, the sum of the terms of this type, i.e.

$$\sum_{i=0}^{n-1}\big[(B_{t_{i+1}}-B_{t_i})^2-h\big]^2 \tag{3.8}$$

for a subdivision $0<t_1<t_2<\dots<t_n$ of stepsize h is a sum of independent random variables of variance $2h^2$. In the independent case, the variance is additive: the variance of the sum is equal to $2nh^2$. Noting that nh is macroscopic, we understand that the action of this term is negligible from a macroscopic point of view.

The reader can check that the argument still holds when the quantity $D^2_{x,x}u(t,B_t)$ is added to sum as in (3.7).

Finally, we write

$$\begin{aligned}
&u(T-(t+h), B_{t+h}-B_t+B_t)\\
&= u(T-t,B_t) - D_t u(t,B_t)h + D_x u(t,B_t)(B_{t+h}-B_t)\\
&\quad + \frac{1}{2}D^2_{x,x}u(t,B_t)h + o(h)\\
&= u(T-t,B_t) + D_x u(t,B_t)(B_{t+h}-B_t) + o(h),
\end{aligned}$$

the second line being obtained by using the PDE. From an infinitesimal point of view (i.e. when getting rid of the negligible terms), we write

$$d\big[u(T-t,B_t)\big] = D_x u(t,B_t)dB_t, \quad 0 \le t \le T, \tag{3.9}$$

We emphasize that the result is not zero! Said differently, the variation of $(u(T-t,B_t))_{0\le t\le T}$ is not zero, as for equations of order one. Actually, understanding $D_x u(t,B_t)dB_t$ as $D_x u(t,B_t)(B_{t+h}-B_t)$, we deduce from the independence of $D_x u(t,B_t)$ and $B_{t+h}-B_t$ that the expectation of the increment is zero. Therefore, $(u(T-t,B_t))_{0\le t\le T}$ is constant... in expectation.

3.4. **Differential Rules.** In the end, everything works as if we had written

$$d\big[u(T-t,B_t)\big] = -D_t u(t,B_t)dt + \frac{1}{2}D^2_{x,x}u(t,B_t)dB_t^2 + D_x u(t,B_t)dB_t,$$

and set $dB_t^2 = dt$. We will use this rule below.

Theorem 3.2. *[Itô's formula] Let $(B_t)_{t\ge 0}$ a real Brownian motion and f a function of class $\mathcal{C}^{1,2}([0,+\infty),\mathbb{R})$. Then, the infinitesimal variation of $(f(t,B_t))_{0\le t\le T}$ writes*

$$d\big[f(t,B_t)\big] = \big[D_t f(t,B_t) + \frac{1}{2}D^2_{x,x}f(t,B_t)\big]dt + D_x f(t,B_t)dB_t.$$

Said differently, Itô's formula is a Taylor formula with convention $dB_t^2 = dt$.

4. Stochastic Integral

We here explain the basic steps of the construction of the stochastic integral. Specifically, the problem is to give a meaning, from a macroscopic point of view, to the term

$$D_x u(t,B_t)dB_t, \tag{4.1}$$

in the statement of Theorem 3.2.

4.1. **Heuristics.** Under a macroscopic form, the term in (4.1) reads as a stochastic integral

$$\int_0^T D_x u(t,B_t)dB_t.$$

This integral is not defined in the Lebesgue sense: Brownian motion paths are not of bounded variation. However, it may be understood in a specific way, as the limit (in a certain sense) of some Riemann sums. Indeed, the integral is understood as the L^2 limit of the sum

$$\sum_{i=0}^{n-1} D_x u(t_i,B_{t_i})\big(B_{t_{i+1}} - B_{t_i}\big),$$

where $0 = t_0 < t_1 < \cdots < t_n$ is a subdivision of $[0,T]$ of (say uniform) stepsize, equal to T/n.

Define now the process (i.e. a family of random variables depending on time)

$$\alpha_t^n = \sum_{i=0}^{n-1} D_x u(t_i,B_{t_i})\mathbf{1}_{(t_i,t_{i+1}]}(t).$$

As a definition of the stochastic integral of such a simple process, we then set

$$\int_0^T \alpha_t^n dB_t := \sum_{i=0}^{n-1} D_x u(t_i, B_{t_i})\big(B_{t_{i+1}} - B_{t_i}\big).$$

As we already said, this term is of zero expectation. The variance is equal to

$$\begin{aligned}
&\mathbb{E}\Big[\Big(\int_0^T \alpha_t^n dB_t\Big)^2\Big] \\
&= \sum_{i=0}^{n-1} \mathbb{E}\big[\big|D_x u(t_i, B_{t_i})\big|^2 |B_{t_{i+1}} - B_{t_i}|^2\big] \\
&\quad + 2 \sum_{0\le i<j\le n-1} \mathbb{E}\big[D_x u(t_i, B_{t_i}) D_x u(t_j, B_{t_j})\big(B_{t_{i+1}} - B_{t_i}\big)\big(B_{t_{j+1}} - B_{t_j}\big)\big].
\end{aligned}$$

In the first sum, we may take advantage of the independence of $B_{t_{i+1}} - B_{t_i}$ and B_{t_i} to split the expectations. Similarly, in the second sum, the expectation of $B_{t_{j+1}} - B_{t_j}$ may be isolated: it is equal to 0. Therefore,

$$\mathbb{E}\Big[\Big(\int_0^T \alpha_t^n dB_t\Big)^2\Big] = h\sum_{i=0}^{n-1} \mathbb{E}\big[\big|D_x u(t_i, B_{t_i})\big|^2\big] = \mathbb{E}\int_0^T (\alpha_t^n)^2 dt.$$

Said differentily, we just built an isometry between $L^2(\Omega, \mathcal{F}, \mathbb{P})$ and $L^2([0,T]\times\Omega, \mathcal{B}(\mathbb{R})\otimes\mathcal{F}, dt\otimes\mathbb{P})$. It is well-seen that the sequence $(\alpha_t^n)_{0\le t\le T}$ converges (at least pointwise) towards $(D_x u(t, B_t))_{0\le t\le T}$. It may be assumed to be bounded if the initial condition u_0 in (3.1) is Lipschitz. Therefore, it has a limit in $L^2([0,T]\times\Omega, \mathcal{B}(\mathbb{R})\otimes\mathcal{F}, dt\otimes\mathbb{P})$ and, thus, is Cauchy. As a consequence, the sequence

$$\Big(\int_0^T \alpha_t^n dB_t\Big)_{0\le t\le T}$$

is Cauchy in $L^2(\Omega, \mathcal{F}, \mathbb{P})$ as well. It is convergent: by definition, the limit is the stochastic integral

$$\int_0^T D_x u(t, B_t) dB_t.$$

4.2. **Construction.** [The reader may skip this part.] Actually, the procedure may be generalized to integrate more general stochastic processes. To do so, we first specify some elements of the theory of stochastic processes (keep in mind that $(\Omega, \mathcal{F}, \mathbb{P})$ stands for a complete probability space):

Definition 4.1. *We call a filtration any non-decreasing family $(\mathcal{F}_t)_{t\ge 0}$ of sub σ-fields of $\mathcal{F}$.*

In practice, a filtration stands for the available information by observation of the events occured between the initial and present times. In what follows, filtrations are assumed to be right-continuous, i.e. $\cap_{\varepsilon>0}\mathcal{F}_{t+\varepsilon} = \mathcal{F}_t$ and complete, i.e. containing sets of zero measure. This is necessary to state some fundamental results for stochastic processes.

Definition 4.2. *A process $(X_t)_{t\ge 0}$ is said to be adapted w.r.t. a filtration $(\mathcal{F}_t)_{t\ge 0}$ if, for any $t \ge 0$, X_t is $\mathcal{F}_t$-measurable. (That is, the value of X_t is known at time t.)*

Definition 4.3. *A Brownian motion $(B_t)_{t\geq 0}$ is said to be an $(\mathcal{F}_t)_{t\geq 0}$-Brownian motion if it is adapted w.r.t. $(\mathcal{F}_t)_{t\geq 0}$ and if, for any $(t,h) \in \mathbb{R}_+^2$, the increment $B_{t+h} - B_t$ is independent of $\mathcal{F}_t$. For instance, a Brownian motion $(B_t)_{t\geq 0}$ is always a Brownian motion w.r.t. its natural filtration*

$$\mathcal{F}_t = \sigma(B_s, s \leq t) \vee \mathcal{N}, \quad t \geq 0. \tag{4.2}$$

Here, $\sigma(B_s, s \leq t)$ stands for the smallest filtration for which the variables $(B_s)_{0\leq s\leq t}$ are measurable and $\mathcal{N}$ for the collection of sets of zero-measure.

We are now in position to generalize the definition of the stochastic integral:

Definition 4.4. *A simple process w.r.t. to the filtration $(\mathcal{F}_t)_{t\geq 0}$ is a process of the form*

$$H_t = \sum_{i=0}^{n-1} H^i \mathbf{1}_{(t_i,t_{i+1}]}(t),$$

where H^i is a square-integrable $\mathcal{F}_{t_i}$-measurable random variable and $0 < t_1 < t_2 < \cdots < t_n$. Then, the stochastic integral is

$$\int_0^{+\infty} H_t dB_t = \sum_{i=0}^{n-1} H^i \big(B_{t_{i+1}} - B_{t_i}\big). \tag{4.3}$$

Using, as above, the independence of H^i and of $B_{t_{i+1}} - B_{t_i}$, we can show that

$$\mathbb{E}\left[\left(\int_0^{+\infty} H_t dB_t\right)^2\right] = \mathbb{E}\int_0^{+\infty} H_t^2 dt.$$

As announced above, the integral defines an isometry. By density, we can extend the definition of the integral to the class of so-called *progressively-measurable processes*:

Definition 4.5. *A process $(H_t)_{t\geq 0}$ is said to be progressively-measurable w.r.t. the filtration $(\mathcal{F}_t)_{t\geq 0}$ if, at any time $T \geq 0$, the joint mapping*

$$(t,\omega) \in [0,T] \times \Omega \mapsto X_t(\omega)$$

is measurable for the product σ-field $\mathcal{B}([0,T]) \otimes \mathcal{F}_T$.

Given a progressively-measurable process such that

$$\mathbb{E}\int_0^{+\infty} H_t^2 dt < +\infty,$$

there exists a sequence $(H_t^n)_{t\geq 0}$ of simple processes converging in $L^2([0,+\infty)\times\Omega, \mathcal{B}([0,+\infty))\otimes \mathcal{F}, dt \otimes \mathbb{P})$ towards $(H_t)_{t\geq 0}$ so that

$$\int_0^{+\infty} H_s dB_s$$

exists as a limit in $L^2(\Omega,\mathcal{F},\mathbb{P})$ of a Cauchy sequence. It satisfies Itô's isometry, i.e.

$$\mathbb{E}\left[\left(\int_0^{+\infty} H_s dB_s\right)^2\right] = \mathbb{E}\int_0^{+\infty} H_s^2 ds.$$

The notion of progressive-measurability is necessary: as the isometry property shows, the process is seen as joint function of time and randomness. As example, it may be proven that any (left- or right-)continuous adapted process is progressively-measurable.

4.3. **Variation of the Integration Bound.** To make the connection between Definition 4.5 and

$$\int_0^T D_x u(t, B_t) dB_t,$$

we understand the above stochastic integral as

$$\int_0^{+\infty} \mathbf{1}_{(0,T]}(t) D_x u(t, B_t) dB_t.$$

Below, we use the first writing only. Going back to (3.9), we finally write (replacing $(B_t)_{t \geq 0}$ by $(x + B_t)_{t \geq 0}$), for all $t \geq 0$,

$$u(T - t, x + B_t) = u(T, x) + \int_0^t D_x u(T - s, x + B_s) dB_s. \tag{4.4}$$

This writing is a bit awkward because of the time reversal. To obtain a straightforward probabilistic formulation, it turns out to be easier to set Eq. (3.1) in a backward sense itself, i.e. with a terminal boundary condition. Actually, in the specific case of Monge-Ampère, this has no real influence since the equation is stationary.

However, we understand from Eq. (4.4) how it may be useful to see the stochastic integral as a process, indexed by the upper integration bound. Actually, it is not so easy to do: the integral being defined as an element of $L^2(\Omega, \mathcal{F}, \mathbb{P})$, it is defined up to an event of zero measure only. To let the upper integration bound vary, it is necessary to choose a suitable version at each time:

Proposition 4.6. *Given a progressively-measurable stochastic process $(H_t)_{t \geq 0}$ w.r.t. a filtration $(\mathcal{F}_t)_{t \geq 0}$ such that*

$$\forall t \geq 0, \quad \mathbb{E} \int_0^t H_s^2 ds < +\infty,$$

it is possible to choose for any $t \geq 0$ a version of the stochastic integral

$$\int_0^t H_s dB_s = \int_0^{+\infty} \mathbf{1}_{]0,t]}(s) H_s dB_s,$$

such that the process

$$\left(\int_0^t H_s dB_s \right)_{t \geq 0}$$

be of continuous paths. (That is, is continuous ω by ω.)

Notice that the continuity property is well-understood in (4.4) since the left-hand side therein is continuous.

4.4. **Martingale Property.** There is another remarkable property of the stochastic integral: it is of zero expectation. Said differently, taking the expectation in (4.4) when $t = T$, we obtain

$$u(T, x) = \mathbb{E}\big[u_0(x + B_T)\big].$$

This is nothing but the representation announced in (3.3): this representation is referred as *Feynman-Kac formula.*

Actually, the centering property for the stochastic integral may be seen as a consequence of a more general property: the stochastic integral is a martingale. The martingale property is a projective property based upon the notion of conditinal expectation:

Definition 4.7. *An adapted process $(M_t)_{t\geq 0}$ w.r.t. a filtration $(\mathcal{F}_t)_{t\geq 0}$ is called a martingale if it is integrable at any time and*

$$\forall 0 \leq s \leq t, \quad \mathbb{E}\big[M_t|\mathcal{F}_s\big] = M_s.$$

In particular, a martingale has a constant expectation.

Go now back to Definition 4.4. Considering (4.3), we notice, with the same notations, that

$$\int_0^{t_j} H_r dB_r = \sum_{i=0}^{j-1} H^i(B_{t_{i+1}} - B_{t_i}),$$

for $0 \leq j \leq n$. By conditioning w.r.t. $\mathcal{F}_{t_{j-1}}$, we obtain

$$\mathbb{E}\bigg[\int_0^{t_j} H_r dB_r|\mathcal{F}_{t_{j-1}}\bigg] = \sum_{i=0}^{j-2} H^i(B_{t_{i+1}} - B_{t_i}) + \mathbb{E}\big[H^{j-1}(B_{t_j} - B_{t_{j-1}})|\mathcal{F}_{t_{j-1}}\big],$$

since the $j-1$ first terms are measurable w.r.t. the σ-field $\mathcal{F}_{t_{j-1}}$. Examinate now the remaining part: we know that H^{j-1} is measurable w.r.t. $\mathcal{F}_{t_{j-1}}$ and that the increment $(B_{t_j} - B_{t_{j-1}})$ is independent of $\mathcal{F}_{t_{j-1}}$. Therefore, the product of both is orthogonal to $L^2(\Omega, \mathcal{F}_{t_{j-1}}, \mathbb{P})$: the conditional expectation is zero. Finally,

$$\mathbb{E}\bigg[\int_0^{t_j} H_r dB_r|\mathcal{F}_{t_{j-1}}\bigg] = \int_0^{t_{j-1}} H_r dB_r.$$

The argument is actually true for any conditioning by $\mathcal{F}_{t_\ell}$, $0 \leq \ell \leq j-1$. Moreover, noting that any pair (s,t), $0 \leq s \leq t$, may be understood as a subset of the subdivision $\{t_0, \dots, t_n\}$, we obtain that

$$\mathbb{E}\bigg[\int_0^{t} H_r dB_r|\mathcal{F}_{s}\bigg] = \int_0^{s} H_r dB_r,$$

for any s and t. By a density argument, we deduce

Proposition 4.8. *Given a progressively-measurable process $(H_t)_{t\geq 0}$ w.r.t. a filtration $(\mathcal{F}_t)_{t\geq 0}$ and satisfying*

$$\forall t \geq 0, \quad \mathbb{E}\bigg[\int_0^t H_s^2 ds\bigg] < +\infty,$$

the stochastic integral

$$\bigg(\int_0^t H_s dB_s\bigg)_{t\geq 0}$$

is a martingale w.r.t. $(\mathcal{F}_t)_{t\geq 0}$.

4.5. **Stopping Times.** The reader may wonder about the connection bewteen a process of zero mean and a martingale. Actually, a martingale is a process whose expectation is zero when stopped at any suitable random times, called *stopping times*.

Here is the definition (together with an example):

Definition 4.9. *Given a filtration $(\mathcal{F}_t)_{t\geq 0}$, a random variable τ with non-negative (but possibly infinite) values is called a stopping-time if*

$$\forall t \geq 0, \quad \{\tau \leq t\} \in \mathcal{F}_t.$$

As an example, a continuous and adapted process $(X_t)_{t\geq 0}$ w.r.t. a filtration $(\mathcal{F}_t)_{t\geq 0}$ and a closed subset $F \subset \mathbb{R}$, the variable

$$\tau := \inf\{t \geq 0 : X_t \in F\},$$

is a stopping time (the infimum being set as $+\infty$ is the set is empty).

Stopping times are really useful because of the following Doob Theorem:

Theorem 4.10. *Given a martingale $(M_t)_{t\geq 0}$ w.r.t. a filtration $(\mathcal{F}_t)_{t\geq 0}$ and a stopping time τ, $(M_{t\wedge\tau})_{t\geq 0}$ is also a martingale (w.r.t. the same filtration). (Here $t \wedge \tau = \min(t, \tau)$.)*

In particular, if τ is bounded by some T, then $\mathbb{E}[M_\tau] = \mathbb{E}[M_{T\wedge\tau}] = \mathbb{E}[M_0]$.

In the above statement, $t \wedge \tau$, for some deterministic time t, is a stopping time again. Indeed, we let the reader check that the minimum of two stopping times is a stopping time as well.

Below, we will also make use of the following version of Doob's theorem:

Theorem 4.11. *For a filtration $(\mathcal{F}_t)_{t\geq 0}$ and a stopping time τ (w.r.t. $(\mathcal{F}_t)_{t\geq 0}$), we call σ-field of events occured before time τ, the σ-field*

$$\mathcal{F}_\tau := \big\{A \in \mathcal{F} : \forall t \geq 0,\ A \cap \{\tau \leq t\} \in \mathcal{F}_t\big\}.$$

Then, for a martingale $(M_t)_{t\geq 0}$ w.r.t. $(\mathcal{F}_t)_{t\geq 0}$ and for another stopping time $\sigma \geq \tau$,

$$\forall t \geq 0, \quad \mathbf{1}_{\{\tau\leq t\}}\mathbb{E}\big[M_{\sigma\wedge t}|\mathcal{F}_\tau\big] = \mathbf{1}_{\{\tau\leq t\}}M_{\sigma\wedge t}.$$

(Again, it is an easy exercice to check that $\{\tau \leq t\}$ is in $\mathcal{F}_\tau$. Indeed, $\mathcal{F}_\tau$ must be understood as the collection of events for which it may be decided if they have occured or not at time τ.)

5. Probabilistic Writing of Monge-Ampère

We now go back to Section 2. In order to give a probabilistic representation of (2.1), we first investigate the probabilistic writing of the solution to the Dirichlet problem

$$\mathrm{Trace}\big[aD^2_{z,\bar{z}}u\big](z) = f(z), \quad z \in \mathcal{D}, \tag{5.1}$$

with the boundary condition $u(z) = g(z)$, $z \in \partial\mathcal{D}$, the non-negative Hermitian matrix a being given.

5.1. **Real Dirichlet Problem.** It may be simpler to start with the real case:

$$\mathrm{Trace}\big[aD^2_{x,x}u\big](x) + f(x) = 0, \quad x \in \mathcal{D} \ ; \quad u(x) = g(x), \quad x \in \partial\mathcal{D},$$

the matrix a being *real*, symmetric and non-negative. Obviously, in this writing, the coefficients f and g together with the domain $\mathcal{D}$ are supposed to be of real structure.

In the case when a is equal to the identity matrix, the process associated with the differential operator $\mathrm{Trace}[D^2_{x,x}\cdot]$ is (up to a multiplicative constant) the d-dimensional Brownian motion, as defined by

Definition 5.1. *A process $(B^1_t, \dots, B^d_t)_{t\geq 0}$ with values in $\mathbb{R}^d$ is called a d-dimensional Brownian motion if each process $(B^i_t)_{t\geq 0}$, $1 \leq i \leq d$, is a Brownian motion and if all of them are independent, i.e., for any time-indices $0 < t_1 < \cdots < t_n$, $n \geq 1$, the vectors $(B^1_{t_1}, \dots, B^1_{t_n})$, $\dots$, $(B^d_{t_1}, \dots, B^d_{t_n})$ are independent.*

Generally speaking, the stochastic integration theory works in dimension d as in dimension 1. Specifically, the point is to consider a common reference filtration: the natural choice consists in replacing B_s in (4.2) by $(B^1_s, \dots, B^d_s)$. It is also necessary to extend the differential rules given in the statement of Theorem 3.2 to the multi-dimensional case.

Theorem 5.2. *Itô's formula (or stochastic Taylor formula) in Theorem 3.2 extends to the multi-dimensional setting. For a d-dimensional Brownian motion $(B_t = (B^1_t, \dots, B^d_t))_{t\geq 0}$ and a function $f \in \mathcal{C}([0, +\infty) \times \mathbb{R}^d, \mathbb{R})$, the infinitesimal variation of $(f(t, B_t))_{t\geq 0}$ expands as*

$$\begin{aligned} &d\big[f(t, B_t)\big] \\ &\quad = \Big[D_t f(t, B_t) + \frac{1}{2}\sum_{i=1}^{d} D^2_{x_i,x_i} f(t, B_t)\Big]dt + \sum_{i=1}^{d} D_{x_i} f(t, B_t) dB^i_t, \quad t \geq 0. \end{aligned}$$

Sketch of the Proof. We just provide the main idea. Generally speaking, the proof relies on the d-dimensional Taylor formula. The only problem is to understand how behave the infinitesimal products $dB^i_t dB^j_t$, $1 \leq i, j \leq d$.

Obviously, $dB^i_t dB^i_t = dt$ for any $1 \leq i \leq d$. When $i \neq j$, $dB^i_t dB^j_t$ is set as 0. This definition may be understood by discretizing the underlying dynamics with a microscopic stepsize. Indeed, if $0 = t_0 < t_1 < \cdots < t_n$ is a time-grid of stepsize h, we may compute

$$\mathbb{E}\Bigg[\bigg(\sum_{k=0}^{n-1}(B^i_{t_{k+1}} - B^i_{t_k})(B^j_{t_{k+1}} - B^j_{t_k})\bigg)^2\Bigg],$$

as in (3.8).

The idea is then the same as in (3.8). Variables are clearly independent and of zero expectation so that the expectation of the square of the sum is equal to the sum of the variances. Now, since $\mathbb{E}[(B^i_{t_{k+1}} - B^i_{t_k})^2(B^j_{t_{k+1}} - B^j_{t_k})^2] = h^2$, the sum is equal to nh^2. It is thus microscopic at the macroscopic level according to the same argument as in (3.8). Macroscopic contributions of the crossed terms are therefore zero. □

We now provide an example of application. (In what follows, we will write B_t for $(B^1_t, \dots, B^d_t)$, so that B_t stands for a vector.)

When $a = (1/2)I_d$ and f and g are regular enough (say f is Hölder continuous and g has Hölder continuous second-order derivatives), it is well-known that the real Dirichlet problem

$$\frac{1}{2}\Delta u(x) + f(x) = 0, \quad x \in \mathcal{D}\ ; \quad u(x) = g(x), \quad x \in \partial\mathcal{D},$$

has a unique classical solution, with bounded derivatives. For $x \in \mathcal{D}$, we write the infinitesimal dynamics of $(u(x + B_t))_{t\geq 0}$. We obtain

$$\begin{aligned} du(x+B_t) &= \sum_{i=1}^{d} D_{x_i}u(x+B_t)dB_t^i + \frac{1}{2}\sum_{i=1}^{d} D^2_{x_i,x_i}u(x+B_t)dt \\ &= \sum_{i=1}^{d} D_{x_i}u(x+B_t)dB_t^i - f(x+B_t)dt. \end{aligned} \tag{5.2}$$

On the macroscopic scale, we obtain (with $B_0 = 0$)

$$u(x+B_t) = u(x) - \int_0^t f(x+B_s)ds + \sum_{i=1}^{d}\int_0^t D_{x_i}u(x+B_t)dB_t^i.$$

This writing is actually unsatisfactory: it holds when $x + B_t$ belongs to $\mathcal{D}$ only; otherwise, it is meaningless. To make things rigorous, we introduce the stopping time:

$$\tau^x := \inf\{t \geq 0 : x + B_t \in \mathcal{D}^{\complement}\}.$$

We are then able to write

$$\begin{aligned} &u(x+B_t) \\ &\quad = u(x) - \int_0^t f(x+B_s)ds + \sum_{i=1}^{d}\int_0^t D_{x_i}u(x+B_t)dB_t^i, \quad 0 \leq t \leq \tau^x. \end{aligned}$$

We emphasize that the martingale term is well-defined since the gradient is bounded. (Atually, for what follows, it would be sufficient that the gradient be continuous inside $\mathcal{D}$ and thus bounded on every compact subset of $\mathcal{D}$.) Taking the expectation at time $t \wedge \tau^x$ and applying Doob's Theorem de Doob 4.10, we obtain

$$\mathbb{E}\big[u(x+B_{t\wedge\tau^x})\big] = u(x) - \mathbb{E}\int_0^{t\wedge\tau^x} f(x+B_s)ds. \tag{5.3}$$

We then intend to let t tend to the infinity. This is possible if $\mathbb{E}[\tau^x] < +\infty$.

Theorem 5.3. *For any $x \in \mathcal{D}$, define τ^x as $\tau^x := \inf\{t \geq 0 : x + B_t \in \mathcal{D}^{\complement}\}$. Then, for any $x \in \mathcal{D}$, $\mathbb{E}[\tau^x] < +\infty$.*

In particular, if f is Hölder continuous on $\mathcal{D}$ and g has Hölder continuous second-order derivatives in the neighborhood of $\bar{\mathcal{D}}$, then the solution u to the Dirichlet problem

$$\frac{1}{2}\Delta u(x) + f(x) = 0, \quad x \in \mathcal{D}\ ; \quad u(x) = g(x), \quad x \in \partial\mathcal{D},$$

admits the following Feynman-Kac representation

$$u(x) = \mathbb{E}\bigg[g(x+B_{\tau^x}) + \int_0^{\tau^x} f(x+B_s)ds\bigg].$$

Proof. It is sufficient to prove $\mathbb{E}[\tau^x] < +\infty$. Feynman-Kac formula then follows by letting t to $+\infty$ in (5.3).

To prove $\mathbb{E}[\tau^x] < +\infty$, we use the non-degeneracy property of the identity matrix in one arbitrarily chosen direction of the space. Compute indeed

$$\begin{aligned} d|x + B_t|^2 &= d\Big[\sum_{i=1}^d |x_i + B_t^i|^2\Big] = \sum_{i=1}^d \big[2(x_i + B_t^i)dB_t^i + (dB_t^i)^2\big] \\ &= 2\sum_{i=1}^d (x_i + B_t^i)dB_t^i + d\,dt. \end{aligned}$$

Take expectation at time $t \wedge \tau^x$. Since $\mathcal{D}$ is bounded, we obtain

$$\sup_{t\geq 0} \mathbb{E}\big[t \wedge \tau^x\big] < +\infty.$$

By monotonous convergence Theorem, we complete the proof. $\square$

When the identity matrix is replaced by a non-zero symmetric matrix a, Brownian motion is replaced by the process

$$X_t := x + \int_0^t \sigma dB_s, \quad t \geq 0, \tag{5.4}$$

where σ is a square-root of a, i.e. $\sigma\sigma^* = a$. This writing must be understood as

$$X_t^i = x_i + \sum_{j=1}^d \int_0^t \sigma_{i,j} dB_s^j, \quad t \geq 0.$$

Following (5.2), we then obtain

$$du(X_t) = \sum_{i=1}^d D_{x_i}u(X_t)dX_t^i + \frac{1}{2}\sum_{i,j=1}^d D^2_{x_i,x_j}u(X_t)dX_t^i dX_t^j, \quad t \geq 0. \tag{5.5}$$

Here, $dX_t^i = \sum_{j=1}^d \sigma_{i,j}dB_t^j$ and the differential rules have the form

$$dX_t^i dX_t^j = \sum_{k,\ell=1}^d \sigma_{i,k}\sigma_{j,\ell}dB_t^k dB_t^\ell = \sum_{k=1}^d \sigma_{i,k}\sigma_{j,k}dt = (\sigma\sigma^*)_{i,j}dt.$$

If $\det(a) \neq 0$, we then obtain an analogous representation to the one obtained for the Laplace operator.

Theorem 5.4. *Consider a positive symmetrix matrix a with σ as square-root, i.e. $a = \sigma\sigma^*$. For any $x \in \mathcal{D}$, consider $(X_t^x)_{t\geq 0}$ as in (5.4) and set $\tau^x := \inf\{t \geq 0 : X_t \in \mathcal{D}^{\complement}\}$. Then, $\mathbb{E}[\tau^x] < +\infty$.*

Moreover, if f is Hölder continuous on $\mathcal{D}$ and g has Hölder continuous second-order derivatives in the neighborhood of $\bar{\mathcal{D}}$, then the solution u to the Dirichlet problem

$$\frac{1}{2}\mathrm{Trace}\big[aD^2_{x,x}u\big](x) + f(x) = 0, \quad x \in \mathcal{D}\ ; \quad u(x) = g(x), \quad x \in \partial\mathcal{D},$$

admits the Feynman-Kac representation

$$u(x) = \mathbb{E}\Big[g(X^x_{\tau^x}) + \int_0^{\tau^x} f(X^x_s)ds\Big].$$

Sketch of the Proof. The boundedness of the expectation of the hitting time is proved as in Theorem 5.3. By Itô's formula (5.5), we complete the proof. □

5.2. **Complex Brownian Motion.** Consider now the complex Dirichlet problem. With the same notation as above (but understood in the complex sense), we are seeking for a representation of the solution u to

$$\mathrm{Trace}\big[aD_{z,\bar{z}}u\big](z) + f(z) = 0, \quad z \in \mathcal{D} \quad ; \quad u(z) = g(z), \quad z \in \partial\mathcal{D}.$$

Here, the matrix a is a non-negative Hermitian matrix.

The solution u may be represented as above. We are going to reproduce the same computations, but with respect to the complex Brownian motion:

Definition 5.5. *A complex Brownian motion of dimension d is a d-dimensional process $(B_t = (B^1_t, \dots, B^d_t))_{t\geq 0}$ with values in $\mathbb{C}^d$ given by*

$$B^j_t = \frac{W^{j,1}_t + \sqrt{-1}\,W^{j,2}_t}{\sqrt{2}}, \quad t \geq 0,\ 1 \leq j \leq d,$$

where the processes $(W^{j,1}_t, W^{j,2}_t)_{1\leq j\leq d}$ are independent real Brownian motions.

We emphasize that the coefficient $\sqrt{2}$ is here to normalize the expectation of the square modulus of B_t, i.e. $\mathbb{E}[|B_t|^2] = t$, $t \geq 0$.

Differential rules are given by

Proposition 5.6. *Let $(B_t = (B^1_t, \dots, B^d_t))_{t\geq 0}$ be a complex Brownian motion of dimension d. Then, Itô's formula in Theorem 5.2 holds with f function of the complex variable of dimension d and with the differential rules*

$$dB^i_t dB^j_t = 0,\ dB^i_t d\bar{B}^j_t = \mathbf{1}_{\{i=j\}}dt, \quad 1 \leq i,j \leq d.$$

Sketch of the Proof. For $1 \leq i \leq d$,

$$dB^i_t dB^i_t = \frac{(dW^{i,1}_t)^2 - (dW^{i,2}_t)^2 + 2\sqrt{-1}\,dW^{i,1}_t dW^{i,2}_t}{2} = 0.$$

Similalry, $d\bar{B}^i_t d\bar{B}^i_t = 0$ and

$$dB^i_t d\bar{B}^i_t = \frac{(dW^{i,1}_t)^2 + (dW^{i,2}_t)^2 + 2\sqrt{-1}\,dW^{i,1}_t dW^{i,2}_t}{2} = dt.$$

Finally, for $1 \leq i < j \leq d$,

$$dB^i_t dB^j_t = dB^i_t d\bar{B}^j_t = 0.$$

This completes the proof. □

Give now several examples.

Example (a). If $d = 1$ and $(Z_t^1)_{t\geq 0}$ and $(Z_t^2)_{t\geq 0}$ admit

$$\begin{aligned} dZ_t^1 &= \sigma_t^1 dB_t + b_t^1 dt \\ dZ_t^2 &= \sigma_t^2 dB_t + b_t^2 dt, \quad t \geq 0, \end{aligned}$$

as dynamics, we obtain

$$\begin{aligned} d(Z_t^1 Z_t^2) &= Z_t^1 dZ_t^2 + Z_t^2 dZ_t^1 + dZ_t^1 dZ_t^2 \\ &= (Z_t^1 \sigma_t^2 + Z_t^2 \sigma_t^1) dB_t + (Z_t^1 b_t^2 + Z_t^2 b_t^1) dt + \sigma_t^1 \sigma_t^2 dB_t dB_t, \quad t \geq 0. \end{aligned}$$

(Pay attention that the absolutely continuous parts $b_t^1 dt$ and $b_t^2 dt$ play no role in the product $dZ_t^1 dZ_t^2$: all the terms they induce are least of order $dt^{3/2}$.) Now, $dB_t dB_t = 0$ in the above equation.

However,

$$\begin{aligned} &d(Z_t^1 \bar{Z}_t^2) \\ &= Z_t^1 d\bar{Z}_t^2 + \bar{Z}_t^2 dZ_t^1 + dZ_1 d\bar{Z}_t^2 \\ &= (Z_t^1 \bar{\sigma}_t^2 d\bar{B}_t + \bar{Z}_t^{\,2} \sigma_t^1 dB_t) + (Z_t^1 \bar{b}_t^2 + \bar{Z}_t^2 b_t^1) dt + \sigma_t^1 \bar{\sigma}_t^2 dB_t d\bar{B}_t, \quad t \geq 0. \end{aligned}$$

Here, $dB_t \cdot d\bar{B}_t = dt$.

In particular, if

$$Z_t = \sum_{j=1}^n \sigma_j dB_t^j, \quad t \geq 0,$$

where $((B_t^j)_{t\geq 0})_j$ are independent complex Brownian motion (i.e. $(B_t = (B_t^1, \dots, B_t^d))_{t\geq 0}$ is a complex Brownian motion of dimension d), then

$$\begin{aligned} d|Z_t|^2 &= Z_t d\bar{Z}_t + \bar{Z}_t dZ_t + dZ_t d\bar{Z}_t \\ &= Z_t \sum_{j=1}^n \bar{\sigma}_j d\bar{B}_t^j + \bar{Z}_t \sum_{j=1}^n \sigma_j dB_t^j + \sum_{j=1}^n \sigma_j \bar{\sigma}_j dt, \quad t \geq 0. \end{aligned}$$

For example, if $\sigma_j = (\sigma\xi)_j$ for a matrix σ, then the last term is equal to $|\sigma\xi|^2$, i.e. to $\langle \bar{\xi}, a\xi \rangle$ where $a = \bar{\sigma}^* \sigma$. This is also equal to $\langle a^* \bar{\xi}, \xi \rangle$.

Example (b). Assume that $d = 1$ and consider an holomorphic function f on $\mathbb{C}$. Then,

$$df(B_t) = f_z'(B_t) dB_t + \frac{1}{2} f_{z,z}''(B_t) dB_t dB_t = f_z'(B_t) dB_t, \quad t \geq 0.$$

In particular, if $\tau_R := \inf\{t \geq 0 : |B_t| \geq R\}$, $R > 0$, then $(f(B_{t\wedge\tau_R}))_{t\geq 0}$ is a martingale. (Here, the stopping time is necessary to guarantee that the martingale is integrable: such an argument is called "a localization argument".) We will say that $(f(B_t))_{t\geq 0}$ is a local martingale.

Example (c). Assume now that $d \geq 1$. Consider a function u with real values of class $\mathcal{C}^2$ on the domain $\mathcal{D}$ and compute $du(X_t)$, $t \geq 0$, where

$$X_t = z + \int_0^t \sigma dB_s, \quad t \geq 0,$$

with σ complex matrix of size $d \times d$. We obtain, for any $t \geq 0$,

$$
\begin{aligned}
&du(X_t) \\
&= \sum_{i=1}^{d} D_{z_i} u(X_t) dX_t^i + \sum_{i=1}^{d} D_{\bar{z}_i} u(X_t) d\bar{X}_t^i \\
&\quad + \frac{1}{2} \sum_{i,j=1}^{d} D^2_{z_i,z_j} u(X_t)(dX_t)^i (dX_t)^j + \frac{1}{2} \sum_{i,j=1}^{d} D^2_{\bar{z}_i,\bar{z}_j} u(X_t)(dX_t)^i (dX_t)^j \\
&\quad + \frac{1}{2} \sum_{i,j=1}^{d} D^2_{z_i,\bar{z}_j} u(X_t)(dX_t)^i (d\bar{X}_t)^j + \frac{1}{2} \sum_{i,j=1}^{d} D^2_{\bar{z}_i,z_j} u(X_t)(d\bar{X}_t)^i (dX_t)^j.
\end{aligned}
$$

It is well-seen that $(dX_t)^i(dX_t)^j = 0$ and $(d\bar{X}_t)^i(d\bar{X}_t)^j = 0$, $1 \leq i,j \leq d$. Moreover, $(dX_t)^i(d\bar{X}_t)^j = \sum_{\ell=1}^{d} \sigma_{i,k}\bar{\sigma}_{k,j} dt = (\sigma\bar{\sigma}^*)_{i,j} dt$. Therefore,

$$
\begin{aligned}
du(X_t) &= \sum_{i=1}^{d} D_{z_i} u(X_t) dX_t^i + \sum_{i=1}^{d} D_{\bar{z}_i} u(X_t) d\bar{X}_t^i \\
&\quad + \frac{1}{2} \mathrm{Trace}\big[a D^2_{z,\bar{z}} u(X_t)\big] dt + \frac{1}{2} \mathrm{Trace}\big[\bar{a} D^2_{\bar{z},z} u(X_t)\big] dt, \quad t \geq 0.
\end{aligned}
$$

Finally, since a and $D^2_{z,\bar{z}} u$ are Hermitian, we deduce

$$
\begin{aligned}
&du(X_t) \\
&= \sum_{i=1}^{d} D_{z_i} u(X_t) dX_t^i + \sum_{i=1}^{d} D_{\bar{z}_i} u(X_t) d\bar{X}_t^i + \mathrm{Trace}\big[a D^2_{z,\bar{z}} u(X_t)\big] dt, \quad t \geq 0.
\end{aligned}
$$

Obviously, this is true for $t \leq \tau^z := \inf\{t \geq 0 : X_t \notin \mathcal{D}\}$ only. We then deduce the analog of Theorem 5.3:

Theorem 5.7. *Let a be a positive Hermitian complex matrix of size $d \times d$ and σ be an Hermitian square-root of a, i.e. $a = \sigma\bar{\sigma}^*$. For a given $z \in \mathcal{D}$ ($\mathcal{D}$ being here assumed to be of the complex variable of dimension d), set*

$$
X_t^z = z + \int_0^t \sigma dB_s, \quad t \geq 0,
$$

together with $\tau^z := \inf\{t \geq 0 : X_t \notin \mathcal{D}\}$. Then, $\mathbb{E}[\tau^z] < +\infty$.

Moreover, for given real-valued functions f and g of the complex variable of dimension d, satisfying the same assumption as in Theorem 5.3, the solution u to the complex Dirichlet problem

$$
\mathrm{Trace}\big[a D^2_{z,\bar{z}} u(z)\big] + f(z) = 0, \quad z \in \mathcal{D} \ ; \quad u(z) = g(z), \quad z \in \partial\mathcal{D},
$$

admits the Feynman-Kac representation

$$
u(z) = \mathbb{E}\bigg[g(X^z_{\tau^z}) + \int_0^{\tau^z} f(X_s^z) ds\bigg].
$$

5.3. **Formulation "*à la Gaveau*".** We are now in position to give a probabilistic representation of the solution of the Monge-Ampère equation. In light of (2.1) and (2.7), a natural candidate to solve the Monge-Ampère equation is

$$\forall z \in \bar{\mathcal{D}}, \quad u(z) = \inf \mathbb{E}\Big[g(X^{\sigma,z}_{\tau^{\sigma,z}}) - \int_0^{\tau^{\sigma,z}} f(X^{\sigma,z}_t)dt\Big], \tag{5.6}$$

the *infimum* being here taken over all progressively-measurable processes $(\sigma_t)_{t\geq 0}$ with values in the set of complex matrices of size d and of determinant of modulus 1, i.e. $\det(\sigma_t \bar{\sigma}^*_t) = 1$ for all $t \geq 0$, with

$$X^{\sigma,z}_t = z + \int_0^t \sigma_s dB_s, \quad t \geq 0\ ; \quad \tau^{\sigma,z} := \inf\{t \geq 0 : X^{\sigma,z}_t \in \mathcal{D}^{\complement}\}. \tag{5.7}$$

We emphasize that this is an *infimum* and not a *supremum* despite the *supremum* in (2.1). The reason may be understood as follows.

Proposition 5.8. *Let σ be a (non-zero) complex matrix of size $d \times d$ and u be a $\mathcal{C}(\bar{\mathcal{D}}) \cap \mathcal{C}^2(\mathcal{D})$ function satisfying*

$$-\mathrm{Trace}\big[aD^2_{z,\bar{z}}u(z)\big] + f(z) \leq 0, \quad z \in \mathcal{D}\ ; \quad u(z) = g(z), \quad z \in \partial\mathcal{D}, \tag{5.8}$$

*where $a = \sigma\sigma^*$ and f and g are functions from $\mathcal{D}$ into $\mathbb{R}$ as in Theorem 5.7 (or as in Assumption (**A**)).*

For a given $z \in \mathcal{D}$, define $(X^z_t)_{t\geq 0}$ and τ^z as in Theorem 5.7. Then,

$$u(z) \leq \mathbb{E}\Big[g(X^z_{\tau^z}) - \int_0^{\tau^z} f(X^z_s)ds\Big].$$

Sketch of the Proof. The proof is similar to the proof of Theorem 5.7 and relies on a simple application of Itô's formula. □

Pay attention that u is here assumed to be smooth. In particular, the reader may object that the solution to the Monge-Ampère equation is not assumed to be of class $\mathcal{C}^2$, so that Proposition 5.8 does not apply to it. Actually, Proposition 5.8 must be understood as some heuristics towards the probabilistic formulation of Monge-Ampère.

In PDE theory, a function u satisfying (5.8) is called a *subsolution* to the Dirichlet problem driven by a, f and g. From a probabilistic point of view, it says that the process $(u(X^z_t))_{t\geq 0}$ is a sub-martingale when $f \geq 0$, i.e. the infinitesimal variation of $(u(X^z_t))_{t\geq 0}$ is greater than the infinitesimal variation of a martingale.

Proposition 5.8 may be seen a variation of the maximum principle: there exists a comparison principle between the solutions of the Dirichlet problems driven by the same matrix a. Going back to the formulation (2.1) of Monge-Ampère, we then understand that the solution to Monge-Ampère is expected to be less than the solution to any Dirichlet problem driven by the same f and g as in Monge-Ampère and by any non-negative Hermitian matrix of determinant 1.

We derive the following representation principle, which may be seen as a probabilistic variation of the Perron-Bremermann method discussed in Chapter 1 by V. Guedj and A. Zeriahi (see Section 1 therein)[2]

[2]We here say "variation" of the Perron-Bremermann method since the optimization below is not performed over a set of plurisubharmonic functions as in the Perron-Bremermann method. Plurisubharmonicity is here

Definition 5.9. *Let f and g be as in Assumption (**A**) and $(B_t)_{t\geq 0}$ be a complex Brownian motion of dimension d. We call* Gaveau representation *or* Gaveau candidate for the Monge-Ampère equation *the function u given by*

$$\forall z \in \bar{\mathcal{D}}, \quad u(z) = \inf \mathbb{E}\left[g\big(X^{\sigma,z}_{\tau^{\sigma,z}}\big) - \int_0^{\tau^{\sigma,z}} f(X^{\sigma,z}_s) ds \right],$$

the infimum *being taken over the set of progressively-measurable processes $(\sigma_t)_{t\geq 0}$ with values in $\mathbb{C}^{d\times d}$ such that* $\det(\sigma_t \bar{\sigma}^*_t) = 1$, $t \geq 0$, *the process $(X^{\sigma,z}_t)_{t\geq 0}$ being given by*

$$X^{\sigma,z}_t = z + \int_0^t \sigma_s dB_s, \quad t \geq 0,$$

and the stopping time $\tau^{\sigma,z}$ by $\tau^{\sigma,z} = \inf\{t \geq 0 : X^{\sigma,z}_t \notin \mathcal{D}\}$.

As the reader may guess, Definition 5.9 goes back to the earlier paper by Gaveau [3]. In fact, it is different from the one used by Krylov in his works and thus different from the one we use below. The reason why Krylov introduced a different representation in his own analysis may be explained as follows: in Definition 5.9, the control σ is poorly controlled! Said differently, the condition on the determinant of $\sigma\bar{\sigma}^*$ is really weak since the norm of the matrix $\sigma\bar{\sigma}^*$ may be as large as possible.

Nevertheless, we emphasize that the connection between the candidate u in Definition 5.9 and the Monge-Ampère equation is rigorously established in the original paper by Gaveau. We refer the reader to it for the complete argument.

5.4. **Krylov Point of View.** Krylov's strategy is a bit different. The starting point consists in writing the original Monge-Ampère formulation

$$\det{}^{1/d}\big[D^2_{z,\bar{z}} u(z)\big] = \frac{1}{d} f(z), \quad z \in \mathcal{D}, \tag{5.9}$$

under the form

$$\sup\big\{-\mathrm{Trace}\big(a D^2_{z,\bar{z}} u(z)\big) + \det{}^{1/d}(a) f(z)\,;\, a = \bar{a}^* \geq 0,\ \mathrm{Trace}(a) = 1\big\} = 0, \tag{5.10}$$

$z \in \mathcal{D}$. Obviously, the first problem is to prove that any $\mathcal{C}^{1,1}$ solution u to (5.10) satisfies (5.9) as well.

Assume therefore that there exists a $\mathcal{C}^{1,1}$ function u from $\mathcal{D}$ to $\mathbb{R}$ solving (5.10) almost everywhere in $\mathcal{D}$. Since u is $\mathcal{C}^{1,1}$, $D^2_{z,\bar{z}} u(z)$ exists for almost every $z \in \mathcal{D}$. By (5.10) and by the sign condition $f \geq 0$, for almost every $z \in \mathcal{D}$, $\mathrm{Trace}(a D^2_{z,\bar{z}} u(z)) \geq 0$ for any non-negative Hermitian matrix a, so that u is plurisubharmonic. Choose now some $z \in \mathcal{D}$ at which $D^2_{z,\bar{z}} u(z)$ exists. If $D^2_{z,\bar{z}} u(z)$ is equal to zero, we can find a positive Hermitian matrix a (with a non-zero determinant) with 1 as trace such that $\mathrm{Trace}(a D^2_{z,\bar{z}} u(z)) = 0$. In particular, (5.10) says that $f(z) \leq 0$ so that $f(z) = 0$ since f is non-negative: (5.9) holds at point z. If the determinant in non-zero at z, the complex Hessian $D^2_{z,\bar{z}} u(z)$ is non-degenerate. In particular it is positive. Therefore, for any sequence $(a_n)_{n\geq 1}$ of non-degenerate matrices approximating the *supremum* in (5.10), the determinant of a_n, $n \geq 1$, is away from zero, uniformly in n. (If the determinant has some vanishing subsequence, we can find a non-zero non-negative Hermitian matrix a such that $\mathrm{Trace}(a D^2_{z,\bar{z}} u(z)) = 0$: by Lemma 2.1, $D^2_{z,\bar{z}} u(z)$ is of zero

hidden in the very large choice for the stochastic process $(\sigma_t)_{t\geq 0}$: this is the reason why we say "probabilistic variation".

determinant.) Therefore, by compactness, there exists a matrix a with 1 as determinant such that

$$-\mathrm{Trace}\big(aD^2_{z,\bar z}u(z)\big) + f(z) = 0.$$

By Lemma 2.1, we understand that $\det^{1/d}(D^2_{z,\bar z}u(z)) \le f(z)/d$. Now, choosing the matrix a in (5.10) as $a = (D^2_{z,\bar z}u(z))^{-1}/\mathrm{Trace}[(D^2_{z,\bar z}u(z))^{-1}]$, we obtain

$$-d + \det^{-1/d}(D^2_{z,\bar z}u(z))f(z) \le 0,$$

i.e. $f(z)/d \le \det^{1/d}(D^2_{z,\bar z}u(z))$, so that equality holds.

The value function associated with the optimal control problem (5.10) admits the following (formal) probabilistic representation

$$\forall z \in \mathcal{D}, \quad u(z) = \inf \mathbb{E}\bigg[g(X^{\sigma,z}_{\tau^{\sigma,z}}) - \int_0^{\tau^{\sigma,z}} \det^{1/d}(\sigma_t\bar\sigma_t^*)f(X_t^{\sigma,z})dt\bigg],$$

the *infimum* being here taken over the progressively-measurable processes $(\sigma_t)_{t\ge0}$ with values in the set of complex matrices of size d such that $\mathrm{Trace}(\sigma_t\bar\sigma_t^*) = 1$ for any $t \ge 0$, with

$$X_t^{\sigma,z} = z + \int_0^t \sigma_s dB_s, \quad t \ge 0\ ; \quad \tau^{\sigma,z} := \inf\{t \ge 0 : X_t^{\sigma,z} \in \mathcal{D}^{\complement}\}.$$

In what follows, we will investigate $-u$ instead of u itself. Changing g into $-g$ in the orginal Monge-Ampère equation, we set

Definition 5.10. *Let f and g be as in Assumption ($\mathbf{A}$) and $(B_t)_{t\ge0}$ be a complex Brownian motion of dimension d. We call* Krylov formulation *of the Monge-Ampère equation driven by the source term f and the boundary condition $-g$ (and not g) the function $-v$, where*

(5.11) $$v(z) = \sup_\sigma v^\sigma(z), \quad z \in \bar{\mathcal{D}},$$

the supremum being here taken over the set of progressively-measurable processes $(\sigma_t)_{t\ge0}$ with values in $\mathbb{C}^{d\times d}$ such $\mathrm{Trace}(\sigma_t\bar\sigma_t^*) = 1$, *$t \ge 0$, and v^σ being given by*

(5.12) $$v^\sigma(z) = \mathbb{E}\bigg[g(X^{\sigma,z}_{\tau^{\sigma,z}}) + \int_0^{\tau^{\sigma,z}} \det^{1/d}(a_t)f(X_t^{\sigma,z})dt\bigg], \quad a_t = \sigma_t\bar\sigma_t^*,$$

the process $(X_t^{\sigma,z})_{t\ge0}$ by

$$X_t^{\sigma,z} = z + \int_0^t \sigma_s dB_s, \quad t \ge 0,$$

and the stopping time $\tau^{\sigma,z}$ by $\tau^{\sigma,z} = \inf\{t \ge 0 : X_t^{\sigma,z} \notin \mathcal{D}\}$.

If v is $\mathcal{C}^{1,1}$ on $\mathcal{D}$ and $-v$ satisfies (5.10) *almost everywhere, i.e.*

(5.13) $$\sup\big\{\mathrm{Trace}\big(aD^2_{z,\bar z}v(z)\big) + \det^{1/d}(a)f(z)\,;\, a = \bar a^* \ge 0,\ \mathrm{Trace}(a) = 1\big\} = 0,$$

a.e. $z \in \mathcal{D}$, then $-v$ is plurisubharmonic and satisfies the Monge-Ampère equation (5.9)*. If $-v$ is continuous up to the boundary $\partial\mathcal{D}$, it admits $-g$ as boundary condition.*

The reader may worry about the boundary condition. First, why is it satisfied? Second, may we expect the solution to be continuous up to the boundary $\partial\mathcal{D}$? The answer to the first question is quite obvious: when $z \in \partial\mathcal{D}$, the stopping time $\tau^{\sigma,z}$ is zero, so that $X^{\sigma,z}_{\tau^{\sigma,z}} = z$. Concerning the second question, we will see below that the answer is clearly positive under Assumption ($\mathbf{A}$).

5.5. **Dynamic Programming Principle.** The Definition 5.10 is not completely satisfactory. The right question is now: may we claim that $-v$ given by (5.11) is a solution to Monge-Ampère without making any reference to the Hamilton-Jacobi-Bellman Equation (5.10)?

We will see below that the answer is almost positive. We say almost because, to say so, we need some regularity property on v, as in Definition 5.10.

Proposition 5.11. *Under the notation of Definition 5.10, assume that the family $(v^\sigma)_\sigma$ is equicontinuous on every compact subset of $\mathcal{D}$ and that v is $\mathcal{C}^{1,1}$ on $\mathcal{D}$. Then, $-v$ satisfies* (5.10) *almost everywhere and thus satisfies the Monge-Ampère equation* (5.9).

Proof. The proof relies on a variation of the so-called "Dynamic Programming Principle" (or Bellman Principle). The main point is to split the cost (5.12) of reaching the boundary of $\mathcal{D}$ when starting from a given point z into two parts: the cost of reaching the boundary of a subdomain from z and the cost of reaching $\partial\mathcal{D}$ when starting from the boundary of the subdomain.

We thus fix a given point $z \in \mathcal{D}$ at which v is twice differentiable in the sense of Taylor, i.e. admits a Taylor expansion of ordrer two at z. (Have in mind that v is almost-everywhere twice differentiable in the sense of Taylor since belongs to $\mathcal{C}^{1,1}(\mathcal{D})$.) Fix also a positive real ε such that the closed (complex) ball $\bar{B}(z,\varepsilon)$ of center z and radius ε is included in $\mathcal{D}$. For any $(\sigma_t)_{t\geq 0}$ as in Definition 5.10, define ρ^σ as the first exit time from the open ball $B(z,\varepsilon)$ by the process $X^{z,\sigma}$, i.e. $\rho^\sigma := \inf\{t \geq 0 : |X_t^{z,\sigma} - z| \geq \varepsilon\}$. Then, the Dynamic Programming Principle writes

Lemma 5.12. *Under the notation of Definition 5.10, assume that the family $(v^\sigma)_\sigma$ is equicontinuous on every compact subset of $\mathcal{D}$. Then, the Dynamic Programming Principle holds in the following way*

$$v(z) = \sup_\sigma \mathbb{E}\left[v(X_{\rho^\sigma}^{\sigma,z}) + \int_0^{\rho^\sigma} \det{}^{1/d}(a_t) f(X_t^{\sigma,z})dt\right], \quad a_t = \sigma_t \bar{\sigma}_t^*, \tag{5.14}$$

the supremum *being here taken w.r.t. the processes $(\sigma_t)_{t\geq 0}$ as in Definition 5.10.*

Proof of the Lower Bound in Lemma 5.12. By (5.12),

$$\begin{aligned} v^\sigma(z) = \mathbb{E}\bigg\{\mathbb{E}\bigg[g(X_{\tau^{\sigma,z}}^{\sigma,z}) + \int_{\rho^\sigma}^{\tau^{\sigma,z}} \det{}^{1/d}(a_t) f(X_t^{\sigma,z})dt | \mathcal{F}_{\rho^\sigma}\bigg] \\ + \int_0^{\rho^\sigma} \det{}^{1/d}(a_t) f(X_t^{\sigma,z})dt\bigg\}. \end{aligned} \tag{5.15}$$

A part of the trick for the Dynamic Programming Principle is the following: the conditional expectation above is less than $v(X_{\rho^\sigma})$. Indeed, for $t \geq \rho^\sigma$,

$$X_t^{\sigma,z} = X_{\rho^\sigma}^{\sigma,z} + \int_{\rho^\sigma}^t \sigma_s dB_s,$$

so that the conditional expectation may be understood as an integration with respect to the trajectories of $(X_t^{\sigma,z})_{t\geq\rho^\sigma}$ with $X_{\rho^\sigma}^{\sigma,z}$ as starting point. (In particular, the interval $[\rho^\sigma, \tau^{\sigma,z}]$ on which $(\det^{1/d}(a_t) f(X_t^{\sigma,z}))_{t\geq 0}$ is integrated in the conditional expectation represents the time passed from ρ^σ up to the exit time from $\mathcal{D}$.) Therefore,

$$v^\sigma(z) \le \mathbb{E}\left[v(X^{\sigma,z}_{\rho^\sigma}) + \int_0^{\rho^\sigma} \det{}^{1/d}(a_t) f(X^{\sigma,z}_t) dt \right]. \tag{5.16}$$

Taking the supremum w.r.t. σ, we complete the proof of the lower bound.

Proof of the Subsolution Property in Monge-Ampère. We now deduce the subsolution property from the lower bound in the Dynamic Programming Principle. Since v is twice Taylor differentiable at z, we can write

$$\begin{aligned} v(X^{\sigma,z}_{\rho^\sigma}) &= v(z) + 2\mathrm{Re}\big[D_z v(z)\big(X^{\sigma,z}_{\rho^\sigma} - z\big)\big] + \frac{1}{2}\big[H^0[v(z)](X^{\sigma,z}_{\rho^\sigma} - z)\big] \\ &\quad + o_\varepsilon(1)\varepsilon^2, \end{aligned} \tag{5.17}$$

the notation $o_\varepsilon(1)$ standing for the Landau notation (i.e. $o_\varepsilon(1)$ tends to 0 with ε) and being independent of the control σ and the underlying randomness ω. Above $H^0[v(z)](\nu)$, for $\nu \in \mathbb{C}^d$, stands for $H^0[v(z)](\nu) = \sum_{i,j=1}^d (D^2_{z_i,z_j} v(z)\nu_i\nu_j + D^2_{z_i,\bar z_j} v(z)\nu_i\bar\nu_j + D^2_{\bar z_i,z_j} v(z)\bar\nu_i\nu_j + D^2_{\bar z_i,\bar z_j} v(z)\bar\nu_i\bar\nu_j)$. By Itô's formula, it is plain to see that

$$\mathbb{E}[H^0[v(z)](X^{\sigma,z}_{\rho^\sigma} - z)] = 2\mathbb{E}\left[\int_0^{\rho^\sigma} \mathrm{Trace}\big(a_t D^2_{z,\bar z} v(z)\big) dt\right].$$

It is also clear that $\mathrm{Re}[D_z v(z)(X^{\sigma,z}_{\rho^\sigma} - z)]$ in (5.17) has zero expectation.

Add now $\int_0^{\rho^\sigma} \det^{1/d}(a_t) f(X^{\sigma,z}_t) dt$ to both sides in (5.17) and take the expectation. Then,

$$\begin{aligned} &\mathbb{E}\left[v(X^{\sigma,z}_{\rho^\sigma}) + \int_0^{\rho^\sigma} \det{}^{1/d}(a_t) f(X^{\sigma,z}_t) dt \right] \\ &= v(z) + \mathbb{E}\left[\int_0^{\rho^\sigma} \big[\mathrm{Trace}\big(a_t D^2_{z,\bar z} v(z)\big) + \det{}^{1/d}(a_t) f(X^{\sigma,z}_t)\big] dt\right] + o_\varepsilon(1)\varepsilon^2. \end{aligned}$$

Therefore, applying (5.16) and using the continuity of f,

$$\begin{aligned} v^\sigma(z) &\le v(z) \\ &\quad + \sup_{a=\bar a^* \ge 0, \mathrm{Trace}(a)=1} \big[\mathrm{Trace}\big(a D^2_{z,\bar z} v(z)\big) + \det{}^{1/d}(a) f(z)\big]\mathbb{E}[\rho^\sigma] \\ &\quad + o_\varepsilon(1)\big(\mathbb{E}[\rho^\sigma] + \varepsilon^2\big). \end{aligned}$$

By Ito's formula, $\varepsilon^2 = \mathbb{E}[|X^\sigma_{\rho^\sigma} - z|^2] = \mathbb{E}[\rho^\sigma]$. Taking the supremum over σ, dividing by ε^2 and letting ε tend to 0, we deduce that

$$\sup_{a=\bar a^* \ge 0, \mathrm{Trace}(a)=1} \left[\mathrm{Trace}\big(a D^2_{z,\bar z} v(z)\big) + \det{}^{1/d}(a) f(z)\right] \ge 0.$$

Proof of the Upper Bound in Lemma 5.12. To prove the supersolution property, we first prove the upper bound in Lemma 5.12. By assumption, we know that the functions $(v^\sigma)_\sigma$ are equicontinuous. Therefore, for a given $\delta > 0$, we can find N points $y_1, \dots, y_N$ on the surface of the ball $B(z,\varepsilon)$ such that, for any $(\sigma_t)_{t \ge 0}$ as above and any $y \in \partial B(z,\varepsilon)$, there exists an index $i(y)$ (say the smallest one) such that $|v^\sigma(y) - v^\sigma(y_{i(y)})| \le \delta$. (Taking the *supremum*, the same holds for v, i.e. $|v(y) - v(y_{i(y)})| \leq \delta$.) Moreover, by definition of the *supremum*, for any index $i \in \{1, \dots, N\}$, we can find a δ-optimal control σ^i such that $v^{\sigma^i}(y_i) + \delta \geq v(y_i) \geq v^{\sigma^i}(y_i)$.

Consider now a control $(\sigma_t)_{t\geq 0}$ of the same type as above. It must be understood as a progressively-measurable functional of the Brownian paths $(B_t)_{t\geq 0}$ and of the (possibly random) initial condition X_0, i.e. something as $(\sigma_t)_{t\geq 0} = (\sigma_t((B_s)_{0\leq s\leq t}, X_0))_{t\geq 0}$. In particular, we emphasize that the value of ρ^σ depends on the values of $(\sigma_t)_{0\leq t<\rho^\sigma}$ only. Moreover, we can modify the values of $(\sigma_t)_{t\geq \rho^\sigma}$ without changing ρ^σ itself. For instance, we can choose σ_t, for $t \geq \rho^\sigma$, as $\sigma_t = \sigma'_{t-\rho^\sigma}((B_{r+\rho^\sigma} - B_{\rho^\sigma})_{0\leq r\leq t-\rho^\sigma}, X^{\sigma,z}_{\rho^\sigma})$ for a new process $(\sigma'_t)_{t\geq 0}$, i.e. we can choose σ_t, for $t \geq \rho^\sigma$, as the new process σ', but shifted in time, the time shift being given by ρ^σ.

For such a choice of $(\sigma_t)_{t\geq 0}$, we are able to compute the conditional expectation in (5.15) explicitly. Indeed, for $(\sigma_t)_{t\geq 0}$ as described above,

$$\begin{aligned}
&\mathbb{E}\Big[g(X^{\sigma,z}_{\tau^{\sigma,z}}) + \int_0^{\tau^{\sigma,z}} \det{}^{1/d}(a_t) f(X^{\sigma,z}_t) dt | \mathcal{F}_{\rho^\sigma}\Big] \\
&= \mathbb{E}\Big[g(X^{\sigma,z}_{\tau^{\sigma,z}}) + \int_{\rho^\sigma}^{\tau^{\sigma,z}} \det{}^{1/d}(a_t) f(X^{\sigma,z}_t) dt | \mathcal{F}_{\rho^\sigma}\Big] \\
&\quad + \int_0^{\rho^\sigma} \det{}^{1/d}(a_t) f(X^{\sigma,z}_t) dt.
\end{aligned} \tag{5.18}$$

Write now $X^{\sigma,z}_t = X^{\sigma,z}_{\rho^\sigma} + \int_{\rho^\sigma}^t \sigma_s dB_s$. Written in a non-rigorous way, this has the form:

$$X^{\sigma,z}_t = X^{\sigma,z}_{\rho^\sigma} + \int_{\rho^\sigma}^t \sigma'_{s-\rho^\sigma}\big((B_{r+\rho^\sigma} - B_{\rho^\sigma})_{0\leq r\leq s}, X^{\sigma,z}_{\rho^\sigma}\big) d\big(B_s - B_{\rho^\sigma}\big).$$

When computing the conditional expectation in the last line of (5.18), everything works as an integration with respect to the trajectories of $(B_t - B_{\rho^\sigma})_{t\geq\rho^\sigma}$: this is a Brownian motion, independent of the past before ρ^σ. Everything thus restarts afresh from $X^{\sigma,z}_{\rho^\sigma}$. Therefore, because of the specific form of σ after ρ^σ (this is the crucial point), the conditional expectation reduces to compute $v^{\sigma'}$ at point $X^{\sigma,z}_{\rho^\sigma}$, so that

$$\begin{aligned}
&\mathbb{E}\Big[g(X^{\sigma,z}_{\tau^{\sigma,z}}) + \int_0^{\tau^{\sigma,z}} \det{}^{1/d}(a_t) f(X^{\sigma,z}_t) dt | \mathcal{F}_{\rho^\sigma}\Big] \\
&\quad = v^{\sigma'}(X^{\sigma,z}_{\rho^\sigma}) + \int_0^{\rho^\sigma} \det{}^{1/d}(a_t) f(X^{\sigma,z}_t) dt.
\end{aligned}$$

Taking the expectation, we deduce a kind of martingale property:

$$v^\sigma(z) = \mathbb{E}\Big[v^{\sigma'}(X^{\sigma,z}_{\rho^\sigma}) + \int_0^{\rho^\sigma} \det{}^{1/d}(a_t) f(X^{\sigma,z}_t) dt\Big]. \tag{5.19}$$

Here is the choice of σ'. Rigoroulsy, we choose σ'_t as $\sigma^{i(X_0)}_t$ where X_0 stands for the (possibly random) initial condition of the process X. Clearly, this means that $\sigma_t = \sigma^{i(X^\sigma_{\rho^{\sigma,z}})}_{t-\rho^\sigma}$, $t > \rho^\sigma$.

For this choice of $(\sigma_t)_{t\geq 0}$, we have from (5.19)

$$
\begin{aligned}
&v(z)\\
&\geq v^{\sigma}(z)\\
&= \mathbb{E}\Big[v^{\sigma'}(X^{\sigma,z}_{\rho^{\sigma}}) + \int_0^{\rho^{\sigma}} \det{}^{1/d}(a_t) f(X^{\sigma,z}_t) dt\Big]\\
&\geq \mathbb{E}\Big[v(X^{\sigma,z}_{\rho^{\sigma}}) + \int_0^{\rho^{\sigma}} \det{}^{1/d}(a_t) f(X^{\sigma,z}_t) dt\Big] - \mathbb{E}\big[|v(X^{\sigma,z}_{\rho^{\sigma}}) - v^{\sigma'}(X^{\sigma,z}_{\rho^{\sigma}})|\big].
\end{aligned}
\tag{5.20}
$$

Now, by the choice of the points $(y_i)_{1\leq i\leq N}$, we know that $|v^{\sigma'}(X^{\sigma,z}_{\rho^{\sigma}}) - v^{\sigma'}(y_{i(X^{\sigma,z}_{\rho^{\sigma}})})| \leq \delta$ and $|v(X^{\sigma,z}_{\rho^{\sigma}}) - v(y_{i(X^{\sigma,z}_{\rho^{\sigma}})})| \leq \delta$. Moreover, by definition, $v^{\sigma'}(y_{i(X^{\sigma,z}_{\rho^{\sigma}})}) = v^{\sigma^j}(y_j)$ with $j = i(X^{\sigma,z}_{\rho^{\sigma}})$ so that $|v^{\sigma'}(y_{i(X^{\sigma,z}_{\rho^{\sigma}})}) - v(y_{i(X^{\sigma,z}_{\rho^{\sigma}})})| \leq \delta$. Therefore

$$
\mathbb{E}\big[|v(X^{\sigma,z}_{\rho^{\sigma}}) - v^{\sigma'}(X^{\sigma,z}_{\rho^{\sigma}})|\big] \leq 3\delta. \tag{5.21}
$$

Plugging (5.21) into (5.20) and letting δ tend to 0, we obtain the upper bound in Lemma 5.12 and thus the equality, i.e. the complete Bellman Principle.

Proof of the Supersolution Property. To deduce the supersolution property in Monge-Ampère, we perform a suitable choice for $(\sigma_t)_{0\leq t\leq \rho^{\sigma}}$ up to time ρ^{σ}. We choose it to be constant between 0 and ρ^{σ}, the constant value being denoted by σ for more simplicity. Expanding $v(X^{\sigma,z}_{\rho^{\sigma,z}})$ in (5.20) as in (5.17) and letting δ and then ε tend to 0, we obtain

$$
\mathrm{Trace}\big(aD^2_{z,\bar{z}}v(z)\big) + \det{}^{1/d}(a) f(z) \leq 0, \quad \text{with } a = \sigma\bar{\sigma}^*.
$$

This completes the proof of Proposition 5.11. □

5.6. **Plurisubharmonicity by Bellman Principle.** We finally emphasize that the Bellman Principle is nothing but a probabilistic version of the plurisubharmonicity property:

Proposition 5.13. *Assume that, for any $z \in \mathcal{D}$, any $\varepsilon > 0$ such that $\bar{B}(z,\varepsilon) \subset \mathcal{D}$ and any $\mathbb{C}^{d\times d}$-valued control $(\sigma_t)_{t\geq 0}$ such that $\mathrm{Trace}(\sigma_t\bar{\sigma}_t^*) = 1$, $t \geq 0$, the process $(X^{\sigma,z}_t)_{t\geq 0}$ given by Definition 5.10 satisfies the Bellman Principle stated in Lemma 5.12 where ρ^{σ} stands therein for the stopping time $\rho^{\sigma} = \inf\{t \geq 0 : |X^{\sigma,z}_t - z| \geq \varepsilon\}$. Assume also that v is continuous on $\mathcal{D}$. Then, v is plurisuperharmonic on $\mathcal{D}$.*

In particular, v is plurisuperharmonic if the family $(v^{\sigma})_{\sigma}$ in Definition 5.10 is equicontinuous on every compact subset of $\mathcal{D}$.

Proof. Given $z \in \mathcal{D}$ and $\varepsilon > 0$ such that $\bar{B}(z,\varepsilon) \subset \mathcal{D}$, it is enough to prove that, for any $\nu \in \mathbb{C}^d$, $|\nu| = 1$,

$$
v(z) \geq \frac{1}{2\pi}\int_0^{2\pi} v\big(z + \varepsilon e^{i\theta}\nu\big) d\theta. \tag{5.22}
$$

In (5.14), we choose σ as the (deterministic) projection matrix on ν, i.e. $\sigma = \nu\bar{\nu}^*$, ν being understood as a column vector. Since f is non-negative, we deduce

$$
v(z) \geq \mathbb{E}\big[v\big(X^{\sigma,z}_{\rho^{\sigma}}\big)\big], \tag{5.23}
$$

with

$$X^{\sigma,z}_{\rho^\sigma} = z + \nu\bar{\nu}^* B_{\rho^\sigma}. \tag{5.24}$$

We now emphasize that $(\bar{\nu}^* B_t)_{t\geq 0}$ is a complex Brownian motion of dimension 1. Indeed, independence of the increments is well-seen and continuity of the trajectories is obviously true as well. It remains to see that $(\mathrm{Re}(\bar{\nu}^* B_t))_{t\geq 0}$ and $(\mathrm{Im}(\bar{\nu}^* B_t))_{t\geq 0}$ are independent non-standard[3] Brownian motions with increments of variance $\Delta/2$ over intervals of length Δ.

Clearly, $\mathrm{Re}(\bar{\nu}^*(B_t - B_s))$, for $0 \leq s \leq t$, is equal to $[\bar{\nu}^*(B_t - B_s) + \nu^*(\bar{B}_t - \bar{B}_s)]/2$. By standard computations, the expectation of the square is equal to $(t-s)/2$, as announced. Similar computations hold for $\mathrm{Im}(\bar{\nu}^*(B_t - B_s))$.

To prove independence, it is sufficient to prove that $\mathrm{Re}(\bar{\nu}^*(B_t - B_s))$ and $\mathrm{Im}(\bar{\nu}^*(B_t - B_s))$ are orthogonal in $L^2(\Omega, \mathbb{P})$ for any $0 \leq s \leq t$[4]. This is easily checked.

Finally, (5.24) yields

$$\varepsilon = |X^{\sigma,z}_{\rho^\sigma} - z| = |\nu\bar{\nu}^* B_{\rho^\sigma}| = |\bar{\nu}^* B_{\rho^\sigma}|,$$

so that ρ^σ stands for the first time when $(\bar{\nu}^* B_t)_{t\geq 0}$ hits the circle of radius ε. By isotropy, the distribution of the hitting point, i.e. $\bar{\nu}^* B_{\rho^\sigma}$, is uniform on the circle. We deduce (5.22) from (5.23). □

6. Program for the Probabilistic Analysis

Krylov's program now consists in establishing

Theorem 6.1. *Let Assumption (**A**) be in force. Then, the value function v in Definition 5.10 belongs to $\mathcal{C}^{1,1}(\bar{\mathcal{D}})$. Moreover, the assumption of Proposition 5.11 is satisfied so that $-v$ solves almost everywhere the Monge-Ampère equation with f as source term and $-g$ as boundary condition.*

As the reader may notice, there are two parts in the statement of Theorem 6.1. The first part must be understood as the main result: it provides the $\mathcal{C}^{1,1}(\bar{\mathcal{D}})$ property for the solution to Monge-Ampère under Assumption (**A**). The second part makes the connection between Krylov's formulation and the original Monge-Ampère equation: the only additional point to prove is the equicontinuity property for the family $(v^\sigma)_\sigma$ on every compact subset of $\mathcal{D}$. Actually, we prove more right below: we prove that equicontinuity holds on the whole $\bar{\mathcal{D}}$ so that v is continuous up to the boundary and satisfies g as boundary condition.

6.1. Equicontinuity of $(v^\sigma)_\sigma$.

We here prove the very first step of our program:

Proposition 6.2. *Under Assumption (**A**) and the notation of Definition 5.10, the functions $(v^\sigma)_\sigma$ are equicontinuous on $\bar{\mathcal{D}}$.*

[3] Non-standard means that the variance of the increments is not normalized.

[4] This argument is false for general processes. It is here true because processes under consideration are of Gaussian type with independent increments. We refer the reader to any lecture on Gaussian vectors and processes.

Proof. We here follow the proof by Gaveau [3]. Below, the control $(\sigma_t)_{t\geq 0}$ is fixed as in Definition 5.10. For given $z, z' \in \mathcal{D}$,

$$\begin{aligned}
&|v^\sigma(z) - v^\sigma(z')| \\
&\leq \mathbb{E}\big[\big|g(X^{\sigma,z}_{\tau^{\sigma,z}}) - g(X^{\sigma,z'}_{\tau^{\sigma,z'}})\big|\big] + \mathbb{E}\int_0^{\tau^{\sigma,z}\wedge\tau^{\sigma,z'}} \big|f(X^{\sigma,z}_s) - f(X^{\sigma,z'}_s)\big| ds \\
&\quad + \mathbb{E}\int_{\tau^{\sigma,z}\wedge\tau^{\sigma,z'}}^{\tau^{\sigma,z}} \big|f(X^{\sigma,z}_s)\big| ds + \mathbb{E}\int_{\tau^{\sigma,z}\wedge\tau^{\sigma,z'}}^{\tau^{\sigma,z'}} \big|f(X^{\sigma,z'}_s)\big| ds.
\end{aligned}$$

(Keep in mind that $\det(a_t) \leq \mathrm{Trace}(a_t) = 1$.) By Assumption (**A**), we can find a constant C, depending on (**A**) only (and whose value may vary from line to line), such that

$$\begin{aligned}
|v^\sigma(z) - v^\sigma(z')| &\leq C\mathbb{E}\big[\big|X^{\sigma,z}_{\tau^{\sigma,z}} - X^{\sigma,z'}_{\tau^{\sigma,z'}}\big|\big] + C\mathbb{E}\int_0^{\tau^{\sigma,z}\wedge\tau^{\sigma,z'}} \big|X^{\sigma,z}_s - X^{\sigma,z'}_s\big| ds \\
&\quad + C\mathbb{E}\big[|\tau^{\sigma,z'} - \tau^{\sigma,z}|\big] \\
&= T_1 + T_2 + T_3.
\end{aligned} \tag{6.1}$$

Above, $a \vee b$ stands for $\max(a,b)$ and $a \wedge b$ for $\min(a,b)$.

To deal with T_2 in (6.1), we emphasize that $X^{\sigma,z}_s - X^{\sigma,z'}_s = z - z'$, $0 \leq s \leq \tau^{\sigma,z} \wedge \tau^{\sigma,z'}$, so that

$$T_2 \leq C|z - z'|\,\mathbb{E}\big[\tau^{\sigma,z}\big].$$

To treat T_1, we notice that

$$\begin{aligned}
\mathbb{E}\big[\big|X^{\sigma,z}_{\tau^{\sigma,z}} - X^{\sigma,z'}_{\tau^{\sigma,z'}}\big|\big] &\leq |z - z'| + \mathbb{E}\bigg[\bigg|\int_{\tau^{\sigma,z}\wedge\tau^{\sigma,z'}}^{\tau^{\sigma,z}\vee\tau^{\sigma,z'}} \sigma_s dB_s\bigg|\bigg] \\
&\leq |z - z'| + \mathbb{E}\bigg[\bigg|\int_{\tau^{\sigma,z}\wedge\tau^{\sigma,z'}}^{\tau^{\sigma,z}\vee\tau^{\sigma,z'}} \sigma_s dB_s\bigg|^2\bigg]^{1/2} \\
&= |z - z'| + \mathbb{E}\bigg[\int_{\tau^{\sigma,z}\wedge\tau^{\sigma,z'}}^{\tau^{\sigma,z}\vee\tau^{\sigma,z'}} \mathrm{Trace}(\sigma_s\bar{\sigma}_s^*) ds\bigg]^{1/2} \\
&= |z - z'| + \mathbb{E}\big[|\tau^{\sigma,z} - \tau^{\sigma,z'}|\big]^{1/2}.
\end{aligned}$$

To complete the proof, it is thus sufficient to prove

Lemma 6.3. *There exists a constant C, depending on (**A**) only, such that for any $z, z' \in \mathcal{D}$, $\mathbb{E}[\tau^{\sigma,z}] \leq C$ and $\mathbb{E}[|\tau^{\sigma,z} - \tau^{\sigma',z}|] \leq C|z - z'|$.*

Proof (Lemma 6.3). Given two different points z and z' in $\mathcal{D}$, we know that $X^{\sigma,z}_t - X^{\sigma,z'}_t = z - z'$ for any $t \leq \tau^{\sigma,z} \wedge \tau^{\sigma,z'}$.

Moreover, on the event $\{\tau^{\sigma,z} \geq \tau^{\sigma,z'}\}$,

$$X^{\sigma,z'}_{\tau^{\sigma,z'}} = X^{\sigma,z'}_{\tau^{\sigma,z'}} - X^{\sigma,z}_{\tau^{\sigma,z'}} + X^{\sigma,z}_{\tau^{\sigma,z'}} = z - z' + X^{\sigma,z}_{\tau^{\sigma,z'}}, \tag{6.2}$$

so that $\mathrm{dist}(X^{\sigma,z}_{\tau^{\sigma,z'}}, \partial\mathcal{D}) \leq |z - z'|$ when $\tau^{\sigma,z} \geq \tau^{\sigma',z}$.

As a consequence, $\mathrm{dist}(X^{\sigma,z}_{\tau^{\sigma,z'}\wedge\tau^{\sigma,z}}, \partial\mathcal{D}) \leq |z - z'|$ on the whole probability space.

Apply now Itô's formula to $(\psi(X_t^{\sigma,z}))_{t\geq 0}$. We obtain

$$\psi\big(X^{\sigma,z}_{\tau^{\sigma,z}}\big) = \psi\big(X^{\sigma,z}_{\tau^{\sigma,z}\wedge\tau^{\sigma,z'}}\big) + \int_{\tau^{\sigma,z}\wedge\tau^{\sigma,z'}}^{\tau^{\sigma,z}} \mathrm{Trace}\big(a_s D^2_{z,\bar z}\psi(X^{\sigma,z}_s)\big)ds$$
$$+ \int_{\tau^{\sigma,z}\wedge\tau^{\sigma,z'}}^{\tau^{\sigma,z}} \big(D_z\psi(X^{\sigma,z}_s)\sigma_s dB_s + D_{\bar z}\psi(X^{\sigma,z}_s)\bar\sigma_s d\bar B_s\big).$$

We emphasize that the LHS is zero. Taking the expectation, we deduce from the plurisuperharmonicity property that

$$\mathbb{E}\big[\psi\big(X^{\sigma,z}_{\tau^{\sigma,z}\wedge\tau^{\sigma,z'}}\big)\big] \geq C\mathbb{E}\big[\tau^{\sigma,z} - \tau^{\sigma,z}\wedge\tau^{\sigma,z'}\big],$$

for some constant $C > 0$ depending on $(\mathbf{A})$ only.

By (6.2), we deduce (C possibly varying from line to line) that

$$\mathbb{E}\big[\big(\tau^{\sigma,z} - \tau^{\sigma,z'}\big)^+\big] = \mathbb{E}\big[\tau^{\sigma,z} - \tau^{\sigma,z}\wedge\tau^{\sigma,z'}\big] \leq C|z-z'|.$$

By symmetry,

$$\mathbb{E}\big[\big|\tau^{\sigma,z} - \tau^{\sigma,z'}\big|\big] \leq C|z-z'|.$$

This completes the proof. □

6.2. **Semi-Convexity Argument.** The main idea to prove the regularity is to reduce the analysis to a convexity problem:

Proposition 6.4. *Assume that the function v is Lipschitz continuous and semi-convex in the whole $\bar{\mathcal{D}}$, i.e. there exists a constant N such that the function $z \in \bar{\mathcal{D}} \mapsto v(z) + N|z|^2$ is convex in any ball included in $\bar{\mathcal{D}}$. Then v belongs to $\mathcal{C}^{1,1}(\bar{\mathcal{D}})$.*

Proof. Proposition 6.4 follows from Lemma 1.3.2 in Chapter 1 by V. Guedj and A. Zeriahi. Indeed, by Proposition 5.13 and Proposition 6.2, $-v$ is plurisubharmonic. Moreover, the semi-convexity property provides the required estimate in Lemma 1.3.2. □

Remark 6.5. *Below, we will also apply Proposition 6.4 on compact subsets of $\mathcal{D}$ (instead of the whole $\bar{\mathcal{D}}$). Obviously, the result then remains true.*

6.3. **Getting Rid of the Supremum.** A very natural idea, to investigate v, is to get rid of, as most as possible, of the *supremum*. In some sense, this is not so difficult since both Lipschitz continuity and (semi-)convexity are stable by *supremum*:

Proposition 6.6. *Let $(w^\beta)_\beta$ be a family of (bounded) functions of the real variable, indexed by some parameter β, for which we can find two functions r_1 and r_2, of the real variable as well, satisfying for any β,*

$$|w^\beta(s) - w^\beta(0)| \leq r_1(s), \quad s \in \mathbb{R},$$

and

$$s \mapsto w^\beta(s) + r_2(s)$$

is convex. Then, the function $s \mapsto \sup_\beta w^\beta(s)$ satisfies the same properties.

The proof is straightforward. The key point is to think of $w^\beta(s)$ as $v^\sigma(\gamma(s))$ for some path $s \in \mathbb{R} \mapsto \gamma(s)$ with values in the domain $\mathcal{D}$, v^σ being given by Definition 5.10. The functions $s \in \mathbb{R} \mapsto r_1(s)$ and $s \in \mathbb{R} \mapsto r_2(s)$ may be understood as $s \in \mathbb{R} \mapsto Ns$ et $s \in \mathbb{R} \mapsto Ns^2$, for some constant N. In such a case, the first inequality in Propostion 6.6 is understood as a Lipschitz property and the second one as a semi-convexity property.

6.4. **Differentiation under the Symbol $\mathbb{E}$.** As we just said, the strategy consists in applying Proposition 6.6 to each function v^σ in Definition 5.10 along a path γ with values in $\mathcal{D}$: this is the way we are able to transfer regularity from the family $(v^\sigma)_\sigma$ to its supremum, i.e. to the function v.

Therefore, the whole problem is now to estimate v^σ uniformly in σ: specifically, we are to estimate the Lipschitz constant and to bound from below the second-order derivatives.

The most natural idea to do so is to differentiate under the symbol $\mathbb{E}$ with respect to the initial condition z in the definition of v^σ, see (5.11), σ being fixed. Remember indeed that the coefficients f and g are differentiable. Remember also that, for each σ, the value $X_t^{\sigma,z}$ of the controlled process at time t is easily differentiable with respect to z, whatever the randomness may be.

Unfortunately, the picture is not so simple. The big deal is the following: the stopping times $\tau^{\sigma,z}$ are not differentiable w.r.t. z.

6.5. **Modification of the Representation.** To be able to differentiate under the symbol $\mathbb{E}$, it is necessary to get rid of the boundary. This means the following: we are to get rid of the boundary condition and to force the representation process to stay in $\mathcal{D}$ forever.

To get rid of the boundary condition, it is sufficient to consider $v^\sigma - g$. Indeed, stochastic differentation rules given in Section 5 show that $v^\sigma - g$ may be written as

$$\big(v^\sigma - g\big)(z) = \mathbb{E}\int_0^{\tau^{\sigma,z}} \big[\det{}^{1/d}(a_t)f(X_t^{\sigma,z}) + \mathrm{Trace}\big(a_t D^2_{z,\bar{z}}g(X_t^{\sigma,z})\big)\big]dt,$$

with $a_t = \sigma_t\bar{\sigma}_t^*$, $t \geq 0$. Obviously, the function g being assumed to be $\mathcal{C}^4$ with bounded derivatives, this operation doesn't modify the regularity property of the second member. However, it may modify its sign.

To recover the right sign, we may use the plurisuperharmonicity condition. Indeed, since

$$\sup_a \sup_{z\in\mathcal{D}} \mathrm{Trace}\big(aD^2_{z,\bar{z}}\psi(z)\big) < 0,$$

(with a as above), we can add $N_0\psi$ to $v^\sigma - g$, for N_0 as large as necessary.

We emphasize that this transform cannot be understood as a modification of the original second member f of the Monge-Ampère equation. Indeed, the coefficients we here remove depend on σ in a more general way than $\det^{1/d}(a_t)f$ does so that the expectation we have to investigate has the form

$$\widetilde{v}^\sigma(z) := \mathbb{E}\int_0^{\tau^{\sigma,z}} F(\det(a_t), a_t, X_t^{\sigma,z})dt, \tag{6.3}$$

which is much more general than the original one in Definition 5.10. We also notice that the general coefficient F is $\mathcal{C}^2$ with respect to the second and third parameters. (Above, $a_t = \sigma_t\bar{\sigma}_t^*$, $t \geq 0$.)

It now remains to get rid of the boundary itself! The idea is to slow down the process $(X_t)_{t\geq 0}$ (forget for the moment the superscripts z and σ to simplify the notations) in the neighborhood of the boundary by means of the function ψ. Consider indeed a stochastic process $(Z_t)_{t\geq 0}$ with the following dynamics:

$$dZ_t = \psi^{1/2}(Z_t)\sigma_t dB_t + a_t D^*_{\bar{z}}\psi(Z_t)dt, \quad t \geq 0, \tag{6.4}$$

and with $Z_0 = z$ as initial condition. Since the dynamics depend on $(Z_t)_{t\geq 0}$ itself, the process $(Z_t)_{t\geq 0}$ is said to satisfy a Stochastic Differential Equation (SDE for short): we give in the next section a short overview of conditions ensuring existence and uniqueness of solutions. Roughly speaking, we will see that the basic conditions are the same as in the theory of Ordinary Differential Equations: Eq. (6.4) is solvable in infinite horizon under global Lipschitz conditions; if the coefficients are locally Lipschitz only on a bounded open subset $\mathcal{U}$, then existence and uniqueness hold up to the first exit time of $\mathcal{U}$. The point is then to discuss whether $(Z_t)_{t\geq 0}$ may reach the boundary of the domain $\mathcal{D}$ or not.

Proposition 6.7. *Given an initial condition $z \in \mathcal{D}$ and a control $(\sigma_t)_{t\geq 0}$ with values in the set of complex matrices of size $d \times d$ such that $\mathrm{Trace}(\sigma_t \bar{\sigma}_t^*) = 1$, $t \geq 0$, the SDE*

$$dZ_t^{\sigma,z} = \psi^{1/2}(Z_t^{\sigma,z})\sigma_t dB_t + a_t D_{\bar{z}}^* \psi(Z_t^{\sigma,z})dt, \quad t \geq 0, \tag{6.5}$$

with the initial condition $Z_0^{\sigma,z} = z$ admits a unique solution. It stays inside $\mathcal{D}$ forever.

Said differently, the stopping time $\tau_\infty^{\sigma,z} := \inf\{t \geq 0 : Z_t^{\sigma,z} \notin \mathcal{D}\}$ (with $\tau_\infty^{\sigma,z} = +\infty$ if the underlying set is empty) is almost-surely infinite.

Proof. The proof relies on a so-called localization argument. For the sake of simplicity, we remove below the superscript (σ, z) in $Z^{\sigma,z}$ and in $\tau_\infty^{\sigma,z}$.

Assume for the moment that (6.5) is indeed solvable. On the interval $[0, \tau_\infty)$, we then compute

$$\begin{aligned} d\psi^{-1}(Z_t) &= -\psi^{-3/2}(Z_t)D_z\psi(Z_t)\sigma_t dB_t - \psi^{-3/2}(Z_t)D_{\bar{z}}\psi(Z_t)\bar{\sigma}_t d\bar{B}_t \\ &\quad - \psi^{-1}(Z_t)\mathrm{Trace}\big[a_t D_{z,\bar{z}}\psi(Z_t)\big]dt, \quad 0 \leq t < \tau_\infty, \end{aligned}$$

with $a_t = \sigma_t \bar{\sigma}_t^*$, $t \geq 0$. Here, the dt term must be understood as

$$\begin{aligned} &- 2\psi^{-2}(Z_t)D_z\psi(Z_t)a_t D_{\bar{z}}^*\psi(Z_t) + \psi(Z_t)\mathrm{Trace}\big[a_t D_{z,\bar{z}}^2\big(\psi^{-1}\big)(Z_t)\big] \\ &= -\psi^{-1}(Z_t)\mathrm{Trace}\big[a_t D_{z,\bar{z}}\psi(Z_t)\big]. \end{aligned}$$

Therefore,

$$\begin{aligned} &d\left[\psi^{-1}(Z_t)\exp\left(\int_0^t \mathrm{Trace}\big[a_s D_{z,\bar{z}}^2\psi(Z_s)\big]ds\right)\right] \\ &= \exp\left(\int_0^t \mathrm{Trace}\big[a_s D_{z,\bar{z}}^2\psi(Z_s)\big]ds\right) \\ &\quad \times \big[-\psi^{-3/2}(Z_t)D_z\psi(Z_t)\sigma_t dB_t - \psi^{-3/2}(Z_t)D_{\bar{z}}\psi(Z_t)d\bar{B}_t\big], \quad 0 \leq t < \tau_\infty. \end{aligned} \tag{6.6}$$

We obtain a (local) martingale.

Indeed, setting $\tau_n := \inf\{t \geq 0 : \psi(Z_t) \leq 1/n\}$, the stochastic integral may be defined rigorously between 0 and τ_n[5]. Therefore, for any $t \geq 0$,

$$\mathbb{E}\left[\psi^{-1}(Z_{t\wedge\tau_n})\exp\left(\int_0^{t\wedge\tau_n} \mathrm{Trace}\big[a_s D_{z,\bar{z}}^2\psi(Z_s)\big]ds\right)\right] = \psi^{-1}(z). \tag{6.7}$$

Noting that $\psi^{-1}(Z_{t\wedge\tau_n}) = n$ if $\tau_n \leq t$, we deduce that, for some constant $C > 0$ independent of n and t,

$$n\exp(-Ct)\mathbb{P}\big\{\tau_n \leq t\big\} \leq \psi^{-1}(z). \tag{6.8}$$

[5]This is the reason why the proof consists of a "localizing" argument.

Thus,
$$\forall n \geq 1,\ t \geq 0, \quad n \exp(-Ct)\mathbb{P}\{\tau_\infty \leq t\} \leq \psi^{-1}(z),$$
since $\tau_\infty \geq \tau_n$. Dividing by n and letting it tend to $+\infty$, we obtain
$$\forall t \geq 0, \quad \mathbb{P}\{\tau_\infty \leq t\} = 0.$$
In particular, $\tau_\infty = +\infty$ almost-surely.

It now remains to prove that both existence and uniqueness hold. Actually, we can solve the truncated version of (6.5)
$$dZ_t^n = \big(\varphi_n \psi^{1/2}\big)(Z_t^n)\sigma_t dB_t + \varphi_n(Z_t^n) a_t D_{\bar{z}}^* \psi(Z_t^n) dt, \quad t \geq 0, \tag{6.9}$$
where φ_n is some smooth cut-off function with values in $[0,1]$ matching 1 on the set $\{\psi \geq 1/n\}$ and 0 on the set $\{\psi \leq 1/(2n)\}$, $n \geq 1$. It is clear that (6.9) is uniquely solvable. (See Subsection 7.1.) Up to the stopping time $\rho_n := \inf\{t \geq 0 : \psi(Z_t^n) \leq 1/n\}$, it satisfies (6.5) as well. In particular, (6.8) holds with ρ_n instead of τ_n, so that $\rho_n \to +\infty$ almost-surely (as $n \to +\infty$). Moreover, by uniqueness of the solution of a Cauchy-Lipschitz SDE, for $m \geq n$, $(Z_t^n)_{t\geq 0}$ and $(Z_t^m)_{t \geq 0}$ are equal up to time $\min(\rho_n, \rho_m) = \rho_n$.

We then set $Z_t = \lim_{n\to+\infty} Z_t^n$. For $t \leq \rho_n$, $n \geq 0$, $Z_t = Z_t^n$ so that $(Z_t)_{0 \leq t \leq \rho_n}$ satisfies (6.5) up to time ρ_n. Letting n tend to $+\infty$, we deduce that $(Z_t)_{t\geq 0}$ satisfies (6.5) over the whole $\mathbb{R}_+$.

Uniqueness follows from the same argument. Any other solution $(Z_t')_{t\geq 0}$ (with the same initial condition) matches $(Z_t)_{t\geq 0}$ up to the first time it exits from $\{\psi \geq 1/n\}$. Letting n tend to $+\infty$, we deduce that there exists a unique solution. $\square$

Obviously, changing $(X_t^{\sigma,z})_{t\geq 0}$ into $(Z_t^{\sigma,z})_{t \geq 0}$ breaks down the representation of v^σ given in Definition 5.10 (and in (6.3)). The point is thus to provide a representation of v (or of $-v$, i.e. of the candidate to solve Monge-Ampère) in terms of the family $((Z_t^{\sigma,z})_{t\geq 0})_\sigma$.

To do so, we first investigate the representation of $\widetilde{v}^\sigma$ when $(\sigma_t)_{t \geq 0}$ is deterministic and constant, i.e. $\sigma_t = \sigma$ deterministic, with $\det(\sigma) \neq 0$.

In the deterministic and constant case, we know that $\widetilde{v}^\sigma$ given in (6.3) satisfies the PDE
$$-\mathrm{Trace}\big[a D_{z,\bar{z}}^2 \widetilde{v}^\sigma(z)\big] = F(\det(a), a, z), \quad z \in \mathcal{D},$$
with zero as boundary condition. (Have in mind that F is here given by adding the $\mathrm{Trace}[aD_{z,\bar{z}}^2(g - N_0\psi)(z)]$ to the original source term $\det^{1/d}(a) f(z)$.)

By Theorem 5.7, we know that $\widetilde{v}^\sigma$ is $\mathcal{C}^2$ inside $\mathcal{D}$ and continuous up to the boundary. In particular, we can apply Itô's formula to $(\psi^{-1}(Z_t^{\sigma,z})\widetilde{v}^\sigma(Z_t^{\sigma,z}))_{t\geq 0}$:

Lemma 6.8. *Under the notation of Proposition 6.7, for any (possibly random) control $(\sigma_t)_{t\geq 0}$ (with values in the set of complex matrices of size $d \times d$ such that $\mathrm{Trace}(\sigma_t \bar{\sigma}_t^*) = 1$, $t \geq 0$) and for any function G in $\mathcal{C}^2(\mathcal{D})$ with real values,*
$$\begin{aligned}
&d\bigg[G(Z_t^{\sigma,z}) \exp\bigg(\int_0^t \mathrm{Trace}\big[a_s D_{z,\bar{z}}\psi(Z_s^{\sigma,z})\big] ds\bigg)\bigg] \\
&= \exp\bigg(\int_0^t \mathrm{Trace}\big[a_s D_{z,\bar{z}}\psi(Z_s^{\sigma,z})\big] ds\bigg)\big[D_z G(Z_t^{\sigma,z})\sigma_t dB_t + D_{\bar{z}} G(Z_t^{\sigma,z})\bar{\sigma}_t d\bar{B}_t\big] \\
&\quad + \exp\bigg(\int_0^t \mathrm{Trace}\big[a_s D_{z,\bar{z}}\psi(Z_s^{\sigma,z})\big] ds\bigg) \mathrm{Trace}\big[a_t D_{z,\bar{z}}(\psi G)(Z_t^{\sigma,z})\big] dt, \quad t \geq 0,
\end{aligned}$$

with $a_t = \sigma_t \bar{\sigma}_t^*$, $t \geq 0$.

In particular, if σ *is constant and non-degenerate, we obtain by choosing* $G = \psi^{-1}\widetilde{v}^\sigma$

$$
\begin{aligned}
&\psi^{-1}(z)\widetilde{v}^\sigma(z) \\
(6.10)\qquad &= \mathbb{E}\int_0^{+\infty} \exp\Big(\int_0^t \mathrm{Trace}\big[aD_{z,\bar{z}}\psi(Z_s^{\sigma,z})\big]ds\Big)F(\det(a), a, Z_t^{\sigma,z})dt, \quad z \in \mathcal{D}.
\end{aligned}
$$

Proof. For simplicity, we get rid of the superscript (σ, z) in $(Z_t^{\sigma,z})_{t\geq 0}$. The first part of the proof is similar to the proof of (6.6). For the second part, it is necessary to localize the dynamics of $(Z_t)_{t\geq 0}$ up to the stopping time $\tau^n = \inf\{t \geq 0 : \psi(Z_t) \leq 1/n\}$ as in (6.7). For $\psi(z) \geq 1/n$, we obtain

$$
\begin{aligned}
\psi^{-1}(z)\widetilde{v}^\sigma(z) &= \mathbb{E}\Big[\exp\Big(\int_0^{t\wedge\tau_n} \mathrm{Trace}\big[aD_{z,\bar{z}}\psi(Z_s)\big]ds\Big)\psi^{-1}(Z_{t\wedge\tau_n})\widetilde{v}^\sigma(Z_{t\wedge\tau_n})\Big] \\
&\quad + \mathbb{E}\int_0^{t\wedge\tau_n} \exp\Big(\int_0^s \mathrm{Trace}\big[aD_{z,\bar{z}}\psi(Z_r)\big]dr\Big)F(\det(a), a, Z_s)ds.
\end{aligned}
$$

We emphasize that the plurisuperharmonicity condition here plays a crucial role: it says that the second integral is exponentially convergent. In particular, the second term in the RHS clearly converges towards the announced quantity as n and t tend to the infinity. The first term in the RHS may be a bit more difficult to handle. By (6.7), we can bound

$$
(6.11)\qquad \mathbb{E}\Big[\exp\Big(\int_0^{t\wedge\tau_n} \mathrm{Trace}\big[aD_{z,\bar{z}}\psi(Z_s)\big]ds\Big)\psi^{-1}(Z_{t\wedge\tau_n})\widetilde{v}^\sigma(Z_{t\wedge\tau_n}); \tau_n \leq t\Big]
$$

by $\psi^{-1}(z)\sup\{\widetilde{v}^\sigma(z'),\ \psi(z') \leq 1/n\}$: this quantity tends to 0 as n tends to the infinity by continuity of $\widetilde{v}^\sigma$ up to the boundary. On the complementary, i.e. on $\{\tau_n > t\}$, we use the plurisuperharmonicity condition to bound (6.11) by $C\exp(-Ct)n$, for a constant C independent of n and t. Letting t tend first to the infinity, and then n, we complete the proof. □

We shall now explain what happens when the control $(\sigma_t)_{t\geq 0}$ in (6.3) and (6.5) is random and evolves with time. Formally, when σ is non-constant, Eq. (6.10) breaks down: the term $\psi^{1/2}$ in Eq. (6.5) is understood as a change of time speed[6] and the process $(Z_t^{\sigma,z})_{t\geq 0}$ appears as a slower version of the original $(X_t^{\sigma,z})_{t\geq 0}$, so that the process $(\sigma_t)_{t\geq 0}$ inside (6.10) cannot be the same as the original one in Eq. (6.3).

The main idea is the following: Eq. (6.10) cannot be a general formula for $\widetilde{v}^\sigma$, but, taking the supremum w.r.t. σ, we recover a representation formula for $\sup_\sigma \widetilde{v}^\sigma$. The idea is not so surprising. Indeed, going back to the proof of the Dynamic Programming Principle, see Lemma 5.12, we understand that the global supremum in (5.11) may be localized, i.e. the values of $(\sigma_t)_{t\geq 0}$ may be locally frozen. Since the representation of $\widetilde{v}^\sigma$ in (6.10) holds for a constant control, we may expect the supremum w.r.t. to (general) σ to satisfy a similar representation formula.

This result turns out to be true: representation (6.10) holds for the value function of the optimization problem. We thus claim

[6] For the reader who knows a bit of stochastic analysis, the drift term in Eq. (6.5) follows from a Girsanov transform.

Proposition 6.9. *Given a control* $(\sigma_t)_{t\geq 0}$ *with values in the set of* $d\times d$ *complex matrices such that* $\mathrm{Trace}(\sigma_t\bar{\sigma}_t^*) = 1$, $t\geq 0$, *consider the function* v^σ *as in Definition 5.10 and modify it into* $\widetilde{v}^\sigma = v^\sigma - g + N_0\psi$ *as in* (6.3) *for some large enough* N_0, *so that*

$$
\begin{aligned}
&\big(\widetilde{v}^\sigma - g + N_0\psi\big)(z) \\
&= \mathbb{E}\int_0^{\tau^{\sigma,z}} \Big[\det{}^{1/d}(a_t) f(X_t^{\sigma,z}) + \mathrm{Trace}\big(a_t D^2_{z,\bar{z}} g(X_t^{\sigma,z})\big) \\
&\qquad\qquad - N_0 \mathrm{Trace}\big(a_t D^2_{z,\bar{z}}\psi(X_t^{\sigma,z})\big)\Big] dt, \\
&:= \mathbb{E}\int_0^{\tau^{\sigma,z}} F(\det(a_t), a_t, X_t^{\sigma,z}) dt, \quad z\in\mathcal{D},
\end{aligned}
$$

with F *non-negative.*

For a given initial condition $z\in\mathcal{D}$, *consider also the SDE*

$$
dZ_t^{\sigma,z} = \psi^{1/2}(Z_t^{\sigma,z})\sigma_t dB_t + a_t D_{\bar{z}}\psi^*(Z_t^{\sigma,z}) dt, \quad t\geq 0, \tag{6.12}
$$

with the initial condition $Z_0^{z,\sigma} = z\in\mathcal{D}$.

Then, the value function $\sup_\sigma[v^\sigma - g + N_0\psi]$ *at point* z *may be expressed as*

$$
v(z) - g(z) + N_0\psi(z) = \sup_\sigma\big[\big(v^\sigma - g + N_0\psi\big)(z)\big] = \psi(z)\sup_\sigma\big[V^\sigma(z)\big],
$$

where

$$
\begin{aligned}
&V^\sigma(z) \\
&= \sup_\sigma \mathbb{E}\bigg[\int_0^{+\infty} \exp\bigg(\int_0^t \mathrm{Trace}\big[a_s D_{z,\bar{z}}\psi(Z_s^{\sigma,z})\big] ds\bigg) F(\det(a_t), a_t, Z_t^{\sigma,z}) dt\bigg],
\end{aligned}
$$

$z\in\mathcal{D}$. *Below, we set* $V(z) := \sup_\sigma V^\sigma(z)$.

7. Derivative Quantity

By Proposition 6.9, we can now forget the boundary constraints. In comparison with the formulation of the complex Monge-Ampère equation given in Section 5, the new representation formula is set in infinite time: we may think of differentiating with respect to the initial condition without taking care of the exit phenomenon.

Unfortunately, there is a price to pay for the new writing. The dynamics of the controlled paths involved in the new representation formula are much less simple to handle with than the original ones. Even without any specific knowledge in stochastic differential equations, it is well-guessed that the derivative of Z in (6.5), if exists, is the solution of a new stochastic differential equation, obtained by differentiation: the whole problem is now to investigate the differentiated equation on the long-run.

7.1. A Word on SDEs.

We said very few about stochastic differential equations. We here specify some elementary facts. (To simplify, things are here stated for real valued processes, but all of them are extendable to the complex case in a standard way.)

A stochastic differential equation may be set in real or complex coordinates. It has the general form

$$
dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t, \quad t\geq 0. \tag{7.1}
$$

Here, the coefficient b is called the *drift* of the equation. It may depend on time, on the solution at current time and on the randomness as well. The same is true for the *diffusion coefficient* σ. Obviously, B here stands for a Brownian motion (with real or complex values according to the framework). We also indicate that the dimension of X may be different from the dimension of B. This is not the case in Proposition 6.9 since the matrix σ is of size $d \times d$. When necessary, we will specify by d the dimension of X and by d_B the dimension of B, so that σ is a matrix of size $d \times d_B$.

Here are the standard solvability conditions. The standard framework for the regularity in space is the Lipschitz one, as we said above: coefficients are assumed to be Lipschitz in space, uniformly in randomness and in time in compact subsets, i.e. $\forall T > 0$, $\exists K_T \geq 0$, $\forall \omega \in \Omega$, $\forall t \in [0,T]$, $\forall x, x'$,

$$|\sigma(t,x) - \sigma(t,x')| + |b(t,x) - b(t,x')| \leq K_T|x - x'|. \tag{7.2}$$

To be sure that the underlying integrals are well-defined, some measurability property is necessary: for any x, the processes $(b(t,x))_{t\geq 0}$ and $(\sigma(t,x))_{t\geq 0}$ are progressively-measurable.

Finally, to control the growth of the coefficients, we ask

$$\forall T \geq 0, \quad \mathbb{E}\int_0^T \big[|b(s,0)|^2 + |\sigma(s,0)|^2\big] ds < +\infty. \tag{7.3}$$

Under these three conditions, existence and uniqueness of a solution to (7.1) with a given initial condition in L^2 hold, on the whole $[0,+\infty)$. The solution has continuous paths that are adapted to the filtration generated by B. Morever, the supremum of the solution is in L^2, locally in time:

$$\forall T \geq 0, \quad \mathbb{E}\big[\sup_{0\leq t\leq T} |X_T|^2\big] < +\infty. \tag{7.4}$$

In the case when the initial condition is in L^p, for some $p > 2$, and (7.3) holds in L^p as well, for the same p, then (7.4) also holds in L^p.

Actually, global Lipschitz conditions may be relaxed. Under local Lipschitz conditions in space, the solution exists on a random interval and may blow up at some random time. As easily-guessed, the blow-up time is a stopping time. It corresponds to the limit of the stopping times ("*first time when the modulus of the solution is larger than m*")$_m$.

Below, we will compare the solutions to stochastic differential equations driven by different coefficients. The following result will be referred to as *a stability property*:

Proposition 7.1. *Consider two sets of coefficients (b,σ) and (b',σ') satisfying (7.2) and (7.3) and denote by $(X_t)_{t\geq 0}$ and $(X'_t)_{t\geq 0}$ the associated solutions for some initial conditions X_0 and X'_0 in L^2. Then, for any $T > 0$, there exists a constant $C_T \geq 0$, only depending on T and K_T, such that, for any event $A \in \mathcal{F}_0$,*

$$\mathbb{E}\big[\mathbf{1}_A \sup_{0\leq t\leq T} |X_t - X'_t|^2\big] \leq C_T\bigg\{\mathbb{E}\big[\mathbf{1}_A|X_0 - X'_0|^2\big] + \mathbb{E}\bigg[\mathbf{1}_A \int_0^T \big(|b - b'|^2(t,X_t) + |\sigma - \sigma'|^2(t,X_t)\big)dt\bigg]\bigg\}.$$

A similar version holds in L^p, for $p > 2$, if the initial conditions are in L^p and (7.3) holds in L^p both for (b,σ) and (b',σ').

(The indicator $\mathbf{1}_A$ here permits to localize the stability property w.r.t. the values of the initial conditions.)

In what follows, the generic equation we consider is of real structure, the complex case being a particular case of the real one by doubling the dimension. The equation is also assumed to be set on the whole space. (Eq. (6.5) is indeed set on the whole space provided ψ be extended to the whole $\mathbb{C}^d$, but the solution stays inside $\mathcal{D}$ forever.)

7.2. **Differentiation of the Flow Generated by a SDE.** Clearly, we have in mind to differentiate under the symbol $\mathbb{E}$ in the representation formula of Proposition 6.9. To do so, we here give some preliminary results about the differentiability of the flow generated by a stochastic differential equation.

Specifically, the following result guarantees the differentiability of the paths $(X_t^x)_{t\geq 0}$ with respect to the starting point x, the coordinates of x being possibly real or complex.

Theorem 7.2. *Assume that, for every $t \geq 0$ and (almost) every $\omega \in \Omega$, the coefficients $b(t,\cdot) : x \in \mathbb{R}^d \mapsto b(t,x)$ and $\sigma(t,\cdot) : x \in \mathbb{R}^d \mapsto \sigma(t,x)$ are of class $\mathcal{C}^3$, with bounded derivatives, uniformly in ω and in t in compact sets. Then, $\mathbb{P}$-almost surely, for all $t \geq 0$, the mapping $x \in \mathbb{R}^d \mapsto X_t^x$ is twice differentiable with respect to x.*

In particular, for any family of initial conditions $(X_0^s)_{s\in\mathbb{R}}$ such that, $\mathbb{P}$-a.s., $s \in \mathbb{R} \mapsto X_0^s$ is $\mathcal{C}^3$, with bounded derivatives, uniformly in ω, the mappings $(s \mapsto X_t^s := X_t^{X_0^s})_{t\geq 0}$ are, $\mathbb{P}$ almost-surely, differentiable with respect to s for all $t \geq 0$. Moreover, $(D_s[X_t^s])_{t\geq 0}$ and $(D_{s,s}^2[X_t^s])_{t\geq 0}$ satisfy linear stochastic differential equations (with random coefficients):

$$\xi_t^s = \gamma'(s) + \int_0^t D_x b(r, X_r^s)\xi_r^s dr + \int_0^t \sum_{j=1}^{d_B} D_x \sigma_{\cdot,j}(r, X_r^s)\xi_r^s dW_r^j, \tag{7.5}$$

and

$$\begin{aligned} \eta_t^s = \gamma''(s) &+ \int_0^t \big[D_x b(r, X_r^s)\eta_r^s + D_{x,x}^2 b(r, X_r^s)\xi_r^s \otimes \xi_r^s\big] dr \\ &+ \int_0^t \sum_{j=1}^{d_B} \big(D_x \sigma_{\cdot,j}(r, X_r^s)\eta_r^s + D_{x,x}^2 \sigma_{\cdot,j}(r, X_r^s)\xi_r^s \otimes \xi_r^s\big) dW_r^j, \end{aligned} \tag{7.6}$$

that is $D_s[X_t^s] = \xi_t^s$ and $D_{s,s}^2[X_t^s] = \eta_t^s$, $t \geq 0$, $s \in \mathbb{R}$.

Proof. We refer the reader to the monograph by Protter [14, Chap. V, Sec. 7, Thm. 39] for the proof.

□

Below, the differentiability property in Theorem 7.2 is referred to as *pathwise twice differentiability*, that is the paths of the process are twice differentiable, randomness by randomness. In some sense, *pathwise differentiability* is too much demanding for our purpose. Indeed, as we recalled above, the point below is to differentiate under the symbol $\mathbb{E}$ only, so that weaker notions of differentiability turn out to be sufficient:

Definition 7.3. *Under the notations of Theorem 7.2, the process $(X_t^s)_{t\geq 0}$ is said to be twice differentiable in probability w.r.t. s if Eqs.* (7.5) *and* (7.6) *are uniquely solvable and, for any*

$T > 0$ *and any* $s \in \mathbb{R}$,

$$(7.7) \qquad \begin{aligned} \forall \nu > 0, \quad & \lim_{\varepsilon \to 0, \varepsilon \neq 0} \mathbb{P}\big\{ \sup_{0 \le t \le T} \big| \delta_\varepsilon X_t^s - \xi_t^s \big| \ge \nu \big\} = 0, \\ & \lim_{\varepsilon \to 0, \varepsilon \neq 0} \mathbb{P}\big\{ \sup_{0 \le t \le T} \big| \delta_\varepsilon \xi_t^s - \eta_t^s \big| \ge \nu \big\} = 0, \end{aligned}$$

with the generic notation $\delta_\varepsilon F_t^s = \varepsilon^{-1}(F_t^{s+\varepsilon} - F_t^s)$ *for some functional* F *depending on* t, s *and possibly* ω.

The process $(X_t^s)_{t \ge 0}$ *is said to be twice differentiable in the mean w.r.t.* s *if Eqs.* (7.5) *and* (7.6) *are uniquely solvable and, for any* $T > 0$ *and any* $s \in \mathbb{R}$,

$$(7.8) \qquad \begin{aligned} \forall p \ge 1, \ & \lim_{\varepsilon \to 0, \varepsilon \neq 0} \mathbb{E}\big[\sup_{0 \le t \le T} \big| \delta_\varepsilon X_t^s - \xi_t^s \big|^p \big] = 0, \\ & \lim_{\varepsilon \to 0, \varepsilon \neq 0} \mathbb{E}\big[\sup_{0 \le t \le T} \big| \delta_\varepsilon \xi_t^s - \eta_t^s \big|^p \big] = 0. \end{aligned}$$

It turns out that *differentiability in the mean* holds under weaker assumptions than *pathwise differentiability*:

Theorem 7.4. *Assume that, for every* $t \ge 0$ *and (almost) every* $\omega \in \Omega$, *the coefficients* $b(t, \cdot) : x \in \mathbb{R}^d \mapsto b(t, x)$ *and* $\sigma(t, \cdot) : x \in \mathbb{R}^d \mapsto \sigma(t, x)$ *are of class* $\mathcal{C}^2$, *with bounded derivatives, uniformly in* t. *Consider a family of initial conditions* $(X_0^s)_{s \in \mathbb{R}}$ *that is twice differentiable in probability, i.e. such that, for any* $s \in \mathbb{R}$,

$$(7.9) \qquad \xi_0^s = \lim_{\varepsilon \to 0, \varepsilon \neq 0} \delta_\varepsilon X_0^s \text{ and } \eta_0^s = \lim_{\varepsilon \to 0, \varepsilon \neq 0} \delta_\varepsilon \xi_0^s$$

exist in probability, i.e. as in (7.7). *Then, the process* $(X_t^s)_{t \ge 0}$ *is twice differentiable in probability w.r.t.* s.

If the random variables $(X_0^s)_{s \in \mathbb{R}}$ *have finite* p*-moments of any order* $p \ge 1$ *and are differentiable in the mean, i.e.* (7.9) *holds as in* (7.8), *then the process* $(X_t^s)_{t \ge 0}$ *is twice differentiable in the mean w.r.t.* s.

The proof is a consequence of the stability property for SDEs. (See Proposition 7.1.)

We now say a word about the connection between the different kinds of differentiability. As easily guessed by the reader, *pathwise differentiability* is stronger than *differentiability in probability*. (This is a straightforward consequence of Lebesgue dominated convergence Theorem. This is also well-understood by comparing the assumptions of Theorems 7.2 and 7.4.) By Markov inequality, it is also clear that *differentiability in the mean* implies *differentiability in probability*.

The converse is true provided some uniform integrability conditions. Consider for example a family of initial conditions $(X_0^s)_{s \in \mathbb{R}}$, with finite p-moments of any order $p \ge 1$, such that the mapping $s \in \mathbb{R} \mapsto X_0^s$ is $\mathcal{C}^3$ almost-surely, with derivatives in any L^p, $p \ge 1$, uniformly in s in compact sets, and assume that, for some stopping τ, $(X_t^s)_{0 \le t \le \tau}$ is *twice differentiable in probability*, uniformly in $t \in [0, \tau]$. (That is T in (7.8) is replaced by τ.) If $\sup_{0 \le t \le \tau} |\xi_t^s|$ and $\sup_{0 \le t \le \tau} |\eta_t^s|$ are in any L^p, $p \ge 1$, uniformly in s in compact sets, then *twice differentiability in the mean* holds uniformly on $[0, \tau]$. As announced, the proof relies on a classical argument in probability theory: convergence in probability implies convergence in any L^p, $p \ge 1$, provided uniform integrability in any L^p, $p \ge 1$. Specifically, the point is to prove that, for any $s \in \mathbb{R}$ and $p \ge 1$, $\sup_{0 \le t \le \tau} |\delta_\varepsilon X_t^s|$ and $\sup_{0 \le t \le \tau} |\delta_\varepsilon \zeta_t^s|$ are in L^p, uniformly in ε in a neighborhood of 0 (ε being different from zero). This may be seen as a consequence of the bounds:

$$
(7.10) \qquad \begin{aligned}
\mathbb{E}\Big[\sup_{0\le t\le \tau} |\delta_\varepsilon X_t^s|^p\Big] &\le \frac{1}{\varepsilon}\int_s^{s+\varepsilon} \mathbb{E}\Big[\sup_{0\le t\le \tau} |\zeta_t^r|^p\Big] dr,\\
\mathbb{E}\Big[\sup_{0\le t\le \tau} |\delta_\varepsilon \zeta_t^s|^p\Big] &\le \frac{1}{\varepsilon}\int_s^{s+\varepsilon} \mathbb{E}\Big[\sup_{0\le t\le \tau} |\eta_t^r|^p\Big] dr,
\end{aligned}
$$

for $\varepsilon > 0$. (Within the framework of Theorem 7.4 and with a similar inequality for $\varepsilon < 0$.) The above inequalities are a straightforward consequence of the first-order Taylor formula when the family $((X_t^s)_{t\ge 0})_{s\in\mathbb{R}}$ is twice differentiable in the pathwise sense, that is when the coefficients b and σ in Theorem 7.2 are smooth. When they are $\mathcal{C}^2$ only, we can approximate them by a sequence of mollified coefficients: by the stability property for SDEs, the derivatives of the solutions to the mollified equations converge towards the derivatives of the true equation; passing to the limit in (7.10), we obtain the expected bounds.

Unless specified, we will work below under the $\mathcal{C}^2$ framework of Theorem 7.4.

7.3. **Derivative Quantity.** In the whole subsection, we choose $X_0^s = \gamma(s)$, γ here standing for a $\mathcal{C}^2$ deterministic curve from $\mathbb{R}$ to $\mathbb{R}^d$, with bounded derivatives. As a consequence of Theorem 7.4, we claim:

Corollary 7.5. *Keep the assumption and notation of Theorem 7.4. Given $T > 0$ and a bounded progressively-measurable random function $f : [0,T]\times\mathbb{R}^d \to \mathbb{R}$ of class $\mathcal{C}^2$ with respect to the spatial parameter and with bounded derivatives, uniformly in time t and in randomness, the real-valued function of the real variable*

$$
s \in [-1,1] \mapsto w_T(s) = \mathbb{E}\int_0^T f(r, X_r^s) dr
$$

admits as first and second-order derivatives:

$$
\begin{aligned}
w_T'(s) &= \mathbb{E}\int_0^T D_x f(r, X_r^s)\xi_r^s dr\\
w_T''(s) &= \mathbb{E}\int_0^T \big(D_x f(r, X_r^s)\eta_r^s + D^2_{x,x} f(r, X_r^s)\xi_r^s \otimes \xi_r^s\big) dr.
\end{aligned}
$$

Corollary 7.5 permits to bound w_T' and w_T''. Indeed, since the equations satisfied by $(\xi_t^s)_{t\ge 0}$ and $(\eta_t^s)_{t\ge 0}$ are linear (with random coefficients), standard stability techniques, based on Gronwall's Lemma, would show that:

$$
(7.11) \qquad \forall p \ge 0, \quad \forall T > 0, \quad \sup_{0\le t\le T} \mathbb{E}\big[|\xi_t^s|^p + |\eta_t^s|^p\big] \le C(p,T),
$$

$C(p,T)$ depending on p, T and the bounds for the derivatives of the coefficients.

Unfortunately, Corollary 7.5 doesn't apply to Proposition 6.9 since T is infinite in Proposition 6.9. Therefore, we must discuss the long-run behavior of $(|\xi_t^s|)_{t\ge 0}$ and $(|\eta_t^s|)_{t\ge 0}$ carefully and, specifically, investigate the long-run integrability against the exponential weight generated by the plurisuperharmonic function ψ, exactly as in the representation formula of Proposition 6.9.

In this framework, we emphasize the following facts. First, in light of Corollary 7.5, it is sufficient to analyze the long-run behavior of the second-order moments of $(|\xi_t^s|)_{t\ge 0}$ and the first-order moments of $(|\eta_t^s|)_{t\geq 0}$. Moreover, the linear structure of $(\eta_t^s)_{t\geq 0}$ being close to the one of $(\xi_t^s)_{t\geq 0}$ (the nonlinear terms in the dynamics of $(|\eta_t^s|)_{t\geq 0}$ being controlled by $(|\xi_t^s|^2)_{t\geq 0}$), it is more or less sufficient to investigate the long-run beahvior of $(|\xi_t^s|^2)_{t\geq 0}$.

Therefore, we now compute the form of $d|\xi_t^s|^2$. Using Itô's formula, we obtain

$$
\begin{aligned}
d|\xi_t^s|^2 &= 2\sum_{i,j=1}^{d_B} (\xi_t^s)^i D_{x_j} b^i(t, X_t^s)(\xi_t^s)^j dt \\
&\quad + \sum_{i=1}^{d}\sum_{j=1}^{n}\left(\sum_{k=1}^{d} D_{x_k}\sigma_{i,j}(t, X_t^s)(\xi_t^s)^k\right)^2 dt + dm_t,
\end{aligned}
\tag{7.12}
$$

dm_t standing for a martingale term, which has no role when computing the expectation. In comparison with Krylov's orginal proof, we emphasize that Krylov makes use of the following shorten notation:

$$
D_\xi b_t^i := \sum_{j=1}^{d} D_{x_j} b^i(t, X_t^s)(\xi_t^s)^j, \quad D_\xi \sigma_t^{i,j} := \sum_{k=1}^{d} D_{x_k}\sigma_{i,j}(t, X_t^s)(\xi_t^s)^k,
$$

so that the dynamics of $|\xi_t^s|^2$ have the form:

$$
d|\xi_t^s|^2 = \left[2\langle \xi_t^s, D_\xi b_t\rangle + |D_\xi \sigma_t|^2\right] dt + dm_t. \tag{7.13}
$$

A typical condition to obtain a long-run control for $(|\xi_t^s|^2)_{t\geq 0}$ is

$$
2\langle \xi_t^s, D_\xi b_t\rangle + |D_\xi \sigma_t|^2 \leq 0, \quad t \geq 0. \tag{7.14}
$$

Indeed, (7.14) implies that $(\mathbb{E}[|\xi_t^s|^2])_{t\geq 0}$ is bounded.

Actually, the reader must understand that the choice we here make is very restrictive: instead of investigating the dynamics of $(|\xi_t^s|^2)_{t\geq 0}$, we could also investigate the dynamics of $(\langle \xi_t^s, A(X_t^s)\xi_t^s\rangle)_{t\geq 0}$ for some smooth function A from $\mathbb{R}^d$ into the set of positive symmetric matrices of dimension d. Indeed, if the spectrum of A is in a compact subset of $(0, +\infty)$, it is equivalent to obtain a long-run control for $(\langle \xi_t^s, A(X_t^s)\xi_t^s\rangle)_{t\geq 0}$ and a long-run control for $(|\xi_t^s|^2)_{t\geq 0}$.

By choosing A possibly different from the identity, we are able to plug some freedom into (7.13) and thus to relax the condition (7.14).

In what follows, we will call:

Definition 7.6. *Under the notation and assumption of Theorem 7.4 and for a smooth function A from $\mathbb{R}^d$ into the set of positive symmetric matrices of size d, we call* derivative quantity *the quadratic process $(\langle A(X_t^s)\xi_t^s, \xi_t^s\rangle)_{t\geq 0}$, denoted by $(\Gamma_t^s)_{t\geq 0}$, and we call* dynamics of the derivative quantity *its absolutely continuous part, denoted by $(\partial\Gamma_t^s)_{t\geq 0}$.*

Specifically, we call dynamics of derivative quantity *(at point $\gamma(s)$) the process (also denoted by $(\partial\Gamma_t(X_t^s, \xi_t^s))_{t\geq 0}$) given by*

$$
\begin{aligned}
\partial\Gamma_t^s &= 2\langle \xi_t^s, A(X_t^s) D_x b(t, X_t^s)\xi_t^s\rangle \\
&\quad + \langle D_x\sigma(t, X_t^s)\xi_t^s, A(X_t^s) D_x\sigma(t, X_t^s)\xi_t^s\rangle \\
&\quad + 2\mathrm{Trace}\left[(D_x\sigma^*(t, X_t^s)\xi_t^s)(D_x A(X_t^s)\xi_t^s)\sigma(t, X_t^s)\right] dt \\
&\quad + \langle \xi_t^s, (L_t A)(X_t^s)\xi_t^s\rangle, \quad t \geq 0,
\end{aligned}
$$

where

$$
\begin{aligned}
&L_t = \sum_{i=1}^{d} b_i(t,\cdot) D_{x_i} + (1/2) \sum_{i,j=1}^{d} (\sigma\sigma^*)_{i,j}(t,\cdot) D^2_{x_i,x_j}\\
&\langle D_x\sigma(t,X_t^s)\xi_t^s, A(X_t^s) D_x\sigma(t,X_t^s)\xi_t^s\rangle\\
&\qquad = \sum_{j=1}^{d_B} \langle D_x\sigma_{\cdot,j}(t,X_t^s)\xi_t^s, A(X_t^s) D_x\sigma_{\cdot,j}(t,X_t^s)\xi_t^s\rangle\\
&\mathrm{Trace}\big[(D_x\sigma^*(t,X_t^s)\xi_t^s)(D_x A(X_t^s)\xi_t^s)\sigma(t,X_t^s)\big]\\
&\qquad = \sum_{i,k=1}^{d} \sum_{j=1}^{d_B} (D_x\sigma_{i,j}(t,X_t^s)\xi_t^s)\big((D_x A_{\cdot,k}(\xi_t^s)_k)\sigma(t,X_t^s)\big)_{i,j}.
\end{aligned}
\tag{7.15}
$$

Following (7.13), *it satisfies*

$$
d\Gamma_t^s = d\langle \xi_t^s, A(X_t^s)\xi_t^s\rangle = \partial\Gamma_t^s dt + dm_t, \quad t \geq 0. \tag{7.16}
$$

(In the complex case, A is an Hermitian functional and Γ_t^s has the form $\langle \xi_t^s, A(X_t^s)\bar{\xi}_t^s\rangle$.)

We claim

Proposition 7.7. *Together with the notations given above, we are also given a real $\delta > 0$ and an $[\delta, +\infty)$-valued (progressively-measurable) random function c both depending on the randomness $\omega \in \Omega$ and on $(t,x) \in [0,+\infty) \times \mathbb{R}^d$ such that, for every $t \geq 0$ and for (almost) every $\omega \in \Omega$, $c(t,\cdot) : x \in \mathbb{R}^d \mapsto c(t,x) \in [\delta,+\infty)$ is of class $\mathcal{C}^2$, with bounded derivatives, uniformly in t and in ω.*

Given an open subset $\mathcal{U} \subset \mathbb{R}^d$ such that $\gamma(s) \in \mathcal{U}$ for some $s \in [-1,1]$, assume that $\partial\Gamma_t^s = \partial\Gamma_t(X_t^s,\xi_t^s) \leq (c(t,X_t^s) - \delta)\Gamma_t^s$ up to the exit time from $\mathcal{U}$, i.e. for $t \leq \tau_U := \inf\{t \geq 0 : X_t^s \notin \mathcal{U}\}$, then, for any $t \geq 0$,

$$
\mathbb{E}\left[\exp\left(-\int_0^{t\wedge\tau_{\mathcal{U}}} (c(r,X_r^s) - \delta)dr\right)\Gamma_{t\wedge\tau_{\mathcal{U}}}\right] \leq \langle \gamma'(s), A(\gamma(s))\gamma'(s)\rangle. \tag{7.17}
$$

Assume for example that $U = \mathbb{R}^d$. Then, with the notation and assumption of Corollary 7.5, there exists a constant C depending on δ and the L^∞ norms (on $\mathcal{U}$) of A^{-1}, $D_x c$, f and $D_x f$ only such that, for any $T > 0$, the function

$$
s \in [-1,1] \mapsto w_T(s) = \mathbb{E}\left[\int_0^T \exp\left(-\int_0^t c(r,X_r^s)dr\right) f(t,X_t^s)dt\right], \tag{7.18}
$$

satisfy $|w_T'(s)| \leq C|\gamma'(s)|$. In particular, the Lipschitz constant of w_T is independent of T.

Proof. The proof is almost straightforward. By (7.16),

$$
\begin{aligned}
&d\left[\exp\left(-\int_0^t \big(c(r,X_r^s) - \delta\big)dr\right)\Gamma_t^s\right]\\
&d\left[\exp\left(-\int_0^t \big(c(r,X_r^s) - \delta\big)dr\right)\langle \xi_t^s, A(X_t^s)\xi_t^s\rangle\right]\\
&= \exp\left(-\int_0^t \big(c(r,X_r^s) - \delta\big)dr\right)\big[\big(\partial\Gamma_t^s - (c(t,X_t^s) - \delta)\Gamma_t^s\big)dt + dm_t\big].
\end{aligned}
$$

Taking the expectation, we get rid of the martingale term. Having, in mind the sign condition on $\partial\Gamma_t^s - (c(t,X_t^s)-\delta)\Gamma_t^s$, we directly deduce (7.17).

To prove the Lipschitz estimate, we first emphasize that, for any $s \in [-1,1]$,

$$
\begin{aligned}
|w_T'(s)| &= \Bigl| \mathbb{E} \int_0^T \exp\Bigl(- \int_0^t c(r,X_r^s) dr \Bigr) \bigl[D_x f(t,X_t^s) \xi_t^s \\
&\qquad\qquad - f(t,X_t^s) \int_0^t D_x c(r,X_r^s) \xi_r^s dr \bigr] \Bigr| \\
&\leq C \mathbb{E} \Bigl[\int_0^T \exp\Bigl(- \int_0^t c(r,X_r^s) dr \Bigr) \bigl[|\xi_t^s| + \int_0^t |\xi_r^s| dr \bigr] dt \Bigr],
\end{aligned}
\tag{7.19}
$$

for some constant C depending on $\|f\|_\infty$, $\|D_x f\|_\infty$ and $\|D_x c\|_\infty$ only.

The result then follows from Lemma 7.8 below. $\square$

Lemma 7.8. *Consider a non-negative process $(c_t)_{t\geq 0}$ together with an $\mathbb{R}^d$-valued process $(\xi_t)_{t\geq 0}$ such that $c_t \geq \delta$, $t \geq 0$, and*

$$
\mathbb{E}\Bigl[\exp\Bigl(- \int_0^t c_r dr \Bigr) |\xi_t|^2 \Bigr] \leq C \exp(-\delta t), \quad t \geq 0,
$$

for some $C \geq 0$ and $\delta > 0$, then

$$
\mathbb{E}\Bigl[\int_0^{+\infty} \exp\Bigl(- \int_0^t c_r dr \Bigr) \Bigl(|\xi_t| + \int_0^t |\xi_r| dr \Bigr) dt \Bigr] \leq C',
$$

for some C' depending on C and δ only.

Proof. From Cauchy-Schwarz inequality and from the bound $c \geq \delta$, we obtain the L^1 version:

$$
\begin{aligned}
\mathbb{E}\Bigl[\exp\Bigl(- \int_0^t c_r dr \Bigr) |\xi_t^s| \Bigr] &\leq \mathbb{E}\Bigl[\exp\Bigl(-2 \int_0^t c_r dr \Bigr) |\xi_t^s|^2 \Bigr]^{1/2} \\
&\leq \exp(-\frac{\delta}{2} t) \mathbb{E}\Bigl[\exp\Bigl(- \int_0^t c_r dr \Bigr) |\xi_t^s|^2 \Bigr]^{1/2} \\
&\leq C^{1/2} \exp(-\delta t), \quad t \geq 0.
\end{aligned}
\tag{7.20}
$$

In particular, since c is always larger than δ, Inequality (7.20) yields

$$
\begin{aligned}
&\mathbb{E}\Bigl[\int_0^{+\infty} \exp\Bigl(- \int_0^t c_r dr \Bigr) \Bigl(|\xi_t| + \int_0^t |\xi_r| dr \Bigr) dt \Bigr] \\
&\leq \mathbb{E}\Bigl[\int_0^{+\infty} \Bigl(\exp\Bigl(- \int_0^t c_r dr \Bigr) |\xi_t| \\
&\qquad + \exp(-\delta t) \int_0^t \exp(\delta r) \exp\Bigl(- \int_0^r c_u du \Bigr) |\xi_r| dr \Bigr) dt \Bigr] \\
&\leq C^{1/2} \int_0^{+\infty} \exp(-\delta t) \bigl(1 + t \bigr) dt.
\end{aligned}
\tag{7.21}
$$

This completes the proof. $\square$

We now perform a similar analysis, but for the second-order derivative $(\langle \eta_t^s, A(X_t^s) \eta_t^s \rangle)_{t\geq 0}$ (see Theorems 7.2 and 7.4) and then for $w_T''(s)$.

Proposition 7.9. *Assume that the assumption of Proposition 7.7 are in force and that σ is bounded. For any $s \in [-1,1]$, denote by $(\Delta_t^s)_{t\geq 0}$ (or by $(\Gamma_t(X_t^s,\eta_t^s))_{t\geq 0}$) the process $(\langle \eta_t^s, A(X_t^s)\eta_t^s\rangle)_{t\geq 0}$ and by $(\partial\Delta_t^s)_{t\geq 0}$ the process*

$$\begin{aligned}
\partial\Delta_t^s &= 2\langle \eta_t^s, A(X_t^s)D_x b(t,X_t^s)\eta_t^s\rangle dt \\
&\quad + \langle D_x\sigma(t,X_t^s)\eta_t^s, A(X_t^s)D_x\sigma(t,X_t^s)\eta_t^s\rangle dt \\
&\quad + 2\mathrm{Trace}\big[(D_x\sigma^*(t,X_t^s)\eta_t^s)(D_x A(X_t^s)\eta_t^s)\sigma(t,X_t^s)\big]dt \\
&\quad + \langle \eta_t^s, (L_t A)(X_t^s)\eta_t^s\rangle, \quad t\geq 0.
\end{aligned}$$

(Be careful that $(\partial\Delta_t^s)_{t\geq 0}$ is not the absolutely continuous part of $(\Delta_t^s)_{t\geq 0}$. It is obtained by replacing $(\xi_t^s)_{t\geq 0}$ by $(\eta_t^s)_{t\geq 0}$ in the definition of $(\partial\Gamma_t^s)_{t\geq 0}$.)

Given an open subset $\mathcal{U}\subset\mathbb{R}^d$ such that $\gamma(s)\in\mathcal{U}$ for some $s\in[-1,1]$, assume that, for all $t\leq \tau_U := \inf\{t\geq 0 : \Gamma_t^s \notin \mathcal{U}\}$, $\partial\Delta_t \leq (c(t,X_t^s)-\delta)\Delta_t$. (Pay attention that this is exactly the same inequality as the one in Proposition 7.7, but with $(\xi_t^s)_{t\geq 0}$ replaced by $(\eta_t^s)_{t\geq 0}$. Clearly, if the one in Proposition 7.7 is true, the current one is expected to be true as well.) Then, there exists a constant C, depending on δ and the L^∞ norms (on $\mathcal{U}$) of A, A^{-1}, D_xA, c, $D^2_{x,x}b$, σ, $D_x\sigma$ and $D^2_{x,x}\sigma$ only, such that, for any $t\geq 0$,

$$\begin{aligned}
&\mathbb{E}\bigg[\exp\bigg(-\int_0^{t\wedge\tau_{\mathcal{U}}}(c(r,X_r^s)-\delta/2)dr\bigg)\big([\Gamma^s_{t\wedge\tau_{\mathcal{U}}}]^2+\Delta^s_{t\wedge\tau_{\mathcal{U}}}\big)^{1/2}\bigg] \\
&\leq \big(\langle\gamma'(s),A(\gamma(s))\gamma'(s)\rangle^2+\langle\gamma''(s),A(\gamma(s))\gamma''(s)\rangle\big)^{1/2} \\
&\quad + C\langle\gamma'(s),A(\gamma(s))\gamma'(s)\rangle.
\end{aligned} \tag{7.22}$$

For example if $\mathcal{U}=\mathbb{R}^d$, the function w_T in (7.18) satisfies

$$|w_T''(s)|\leq \frac{C}{(1\wedge\delta)^3}\big(|\gamma'(s)|^2+|\gamma''(s)|\big),\quad s\in[-1,1].$$

for a possible modified value of the constant C, depending on the L^∞ norms (on $\mathcal{U}$) of D_xc, $D^2_{x,x}c$, f, D_xf and $D^2_{x,x}f$ as well. (In particular, it is independent of T and s.)

Proof. For simplicity, we make use of Krylov's notations, i.e. we set: $D_\eta b_t^s := D_x b(t,X_t^s)\eta_t^s$, $D^2_{\xi,\xi}b_t^s := D^2_{x,x}b(t,X_t^s)\xi_t^s\otimes\xi_t^s$, $D_\eta(\sigma_t^s)_{\cdot,j} := D_x\sigma_{\cdot,j}(t,X_t^s)\eta_t^s$ and finally $D^2_{\xi,\xi}(\sigma_t^s)_{\cdot,j} := D^2_{x,x}\sigma_{\cdot,j}(t,X_t^s)\xi_t^s\otimes\xi_t^s$. With these notations, η in Theorem 7.2 has the form:

$$d\eta_t^s = D_\eta b_t^s dt + D^2_{\xi,\xi}b_t^s dt + \sum_{j=1}^{d_B} D_\eta(\sigma_t^s)_{\cdot,j}dW_t^j + \sum_{j=1}^{d_B} D^2_{\xi,\xi}(\sigma_t^s)_{\cdot,j}dW_t^j,$$

$t\geq 0$. Considering the quadratic form driven by A, we obtain (with the notation $A_t^s = A(X_t^s)$)

$$\begin{aligned}
&d\langle\eta_t^s, A_t^s\eta_t^s\rangle \\
&= 2\langle\eta_t^s, A_t^s D_\eta b_t^s\rangle dt + 2\langle\eta_t^s, A_t^s D^2_{\xi,\xi}b_t^s\rangle dt \\
&\quad + 2\mathrm{Trace}\big[(D_\eta\sigma^*(t,X_t^s)+D^2_{\xi,\xi}\sigma^*(t,X_t^s))(D_xA(X_t^s)\eta_t^s)\sigma(t,X_t^s)\big]dt \\
&\quad + \langle\big(D_\eta\sigma_t^s+D^2_{\xi,\xi}\sigma_t^s\big), A_t^s\big(D_\eta\sigma_t^s+D^2_{\xi,\xi}\sigma_t^s\big)\rangle dt + \langle\eta_t^s, L_tA_t^s\eta_t^s\rangle dt + dm_t,
\end{aligned}$$

$t\geq 0$, $(m_t)_{t\geq 0}$ standing for a generic martingale term that is (more or less) useless in what follows. (See (7.15) for the definition of $\langle\big(D_\eta\sigma_t^s+D^2_{\xi,\xi}\sigma_t^s\big), A_t^s\big(D_\eta\sigma_t^s+D^2_{\xi,\xi}\sigma_t^s\big)\rangle$.) Following the proof of (7.16),

$$d\big[\langle \xi_t^s, A_t^s \xi_t^s\rangle^2\big] = 2\langle \xi_t^s, A_t^s \xi_t^s\rangle \partial \Gamma_t^s dt + \big|2A_t^s D_\xi \sigma_t^s \xi_t^s + \langle \xi_t^s, D_\sigma A(X_t^s)\xi_t^s\rangle\big|^2 dt + dm_t.$$

(Here again, the generic notation $(m_t)_{t\geq 0}$ stands for a martingale. Morever, the term $|2A_t^s D_\xi \sigma_t^s \xi_t^s + \langle \xi_t^s, D_\sigma A(X_t^s)\xi_t^s\rangle|^2$ stands for $\sum_{j=1}^{d_B} |2\langle A_t^s D_\xi(\sigma_t^s)_{\cdot,j}, \xi_t^s\rangle + \sum_{k=1}^{d} \langle \xi_t^s, D_{x_k} A(X_t^s)\xi_t^s\rangle \sigma_{k,j}(t, X_t^s)|^2$.) Therefore,

$$\begin{aligned}
&d\big(\langle \xi_t^s, A_t^s \xi_t^s\rangle^2 + \langle \eta_t^s, A_t^s \eta_t^s\rangle\big)\\
&= 2\langle \eta_t^s, A_t^s D_\eta b_t^s\rangle dt + 2\langle \eta_t^s, A_t^s D_{\xi,\xi}^2 b_t^s\rangle dt + \langle \eta_t^s, L_t A_t^s \eta_t^s\rangle dt\\
&\quad + 2\mathrm{Trace}\big[(D_\eta \sigma^*(t, X_t^s) + D_{\xi,\xi}^2 \sigma^*(t, X_t^s))(D_x A(X_t^s)\eta_t^s)\sigma(t, X_t^s)\big] dt\\
&\quad + \langle \big(D_\eta \sigma_t^s + D_{\xi,\xi}^2 \sigma_t^s\big), A_t^s\big(D_\eta \sigma_t^s + D_{\xi,\xi}^2 \sigma_t^s\big)\rangle dt + 2\langle \xi_t^s, A_t^s \xi_t^s\rangle \partial \Gamma_t^s dt\\
&\quad + \big|2A_t^s D_\xi \sigma_t^s \xi_t^s + \langle \xi_t^s, D_\sigma A(X_t^s)\xi_t^s\rangle\big|^2 dt + dm_t.
\end{aligned}$$

Apply now the function $x \in \mathbb{R} \mapsto (a + x)^{1/2}$, for some small $a > 0$. It is a concave function, so that the second-order term deriving from Itô's formula is non-increasing. In particular, we write (in a little bit crude way)

$$\begin{aligned}
&d\big(a + \langle \xi_t^s, A_t^s \xi_t\rangle^2 + \langle \eta_t^s, A_t^s \eta_t^s\rangle\big)^{1/2}\\
&\leq \frac{1}{2}\big(a + \langle \xi_t^s, A_t^s \xi_t\rangle^2 + \langle \eta_t^s, A_t^s \eta_t^s\rangle\big)^{-1/2}\big[2\langle \eta_t^s, A_t^s D_\eta b_t^s\rangle + 2\langle \eta_t^s, A_t^s D_{\xi,\xi}^2 b_t^s\rangle\\
&\quad + 2\mathrm{Trace}\big[(D_\eta \sigma^*(t, X_t^s) + D_{\xi,\xi}^2 \sigma^*(t, X_t^s))(D_x A(X_t^s)\eta_t^s)\sigma(t, X_t^s)\big] dt\\
&\quad + \langle \eta_t^s, L_t A_t^s \eta_t^s\rangle + \langle \big(D_\eta \sigma_t^s + D_{\xi,\xi}^2 \sigma_t^s\big), A_t^s\big(D_\eta \sigma_t^s + D_{\xi,\xi}^2 \sigma_t^s\big)\rangle\\
&\quad + 2\langle \xi_t^s, A_t^s \xi_t^s\rangle \partial \Gamma_t^s dt + \big|2A_t^s D_\xi \sigma_t^s \xi_t^s + \langle \xi_t^s, D_\sigma A(X_t^s)\xi_t^s\rangle\big|^2\big] + dm_t.
\end{aligned} \tag{7.23}$$

We now claim that

$$\begin{aligned}
&2\langle \eta_t^s, A_t^s D_\eta b_t^s\rangle + \langle \eta_t^s, L_t A_t^s \eta_t^s\rangle + \langle \big(D_\eta \sigma_t^s + D_{\xi,\xi}^2 \sigma_t^s\big), A_t^s\big(D_\eta \sigma_t^s + D_{\xi,\xi}^2 \sigma_t^s\big)\rangle\\
&\quad + 2\mathrm{Trace}\big[(D_\eta \sigma^*(t, X_t^s) + D_{\xi,\xi}^2 \sigma^*(t, X_t^s))(D_x A(X_t^s)\eta_t^s)\sigma(t, X_t^s)\big]\\
&= \partial \Delta_t^s + 2\langle D_\eta \sigma_t^s, A_t^s D_{\xi,\xi}^2 \sigma_t^s\rangle + \langle D_{\xi,\xi}^2 \sigma_t^s, A_t^s D_{\xi,\xi}^2 \sigma_t^s\rangle\\
&\quad + 2\mathrm{Trace}\big[D_{\xi,\xi}^2 \sigma^*(t, X_t^s)(D_x A(X_t^s)\eta_t^s)\sigma(t, X_t^s)\big]\\
&= \partial \Delta_t^s + O\big((a + |\xi_t^s|^4 + |\eta_t^s|^2)^{1/2}|\xi_t^s|^2\big),
\end{aligned}$$

the notation $O(\dots)$ standing for the Landau notation. Here, we emphasize that the underlying constant in $O(\cdots)$ depends on the L^∞ norms (on $\mathcal{U}$) of A, $D_x A$, σ, $D_x \sigma$ and $D_{x,x}^2 \sigma$ only and, in particular, is independent of t and ω. Actually, all the remaining terms in (7.23) except the martingale term can be bounded by $O((a + |\xi_t^s|^4 + |\eta_t^s|^2)^{1/2}|\xi_t^s|^2)$ as well, the underlying constant in $O(\cdots)$ possibly depending on the L^∞ norms (on $\mathcal{U}$) of A^{-1}, c and $D_{x,x}^2 b$ also. Therefore, we can find some constant $C > 0$, depending on the L^∞ norms (on $\mathcal{U}$) of A, A^{-1}, $D_x A$, c, $D_{x,x}^2 b$, σ, $D_x \sigma$ and $D_{x,x}^2 \sigma$ only, such that

$$\begin{aligned}
&d\big(a + \langle \xi_t^s, A_t^s \xi_t^s\rangle^2 + \langle \eta_t^s, A_t^s \eta_t^s\rangle\big)^{1/2}\\
&\quad \leq \frac{1}{2}\big(a + \langle \xi_t^s, A_t^s \xi_t^s\rangle^2 + \langle \eta_t^s, A_t^s \eta_t^s\rangle\big)^{-1/2}\partial \Delta_t^s dt + C|\xi_t^s|^2 dt + dm_t.
\end{aligned}$$

Finally, following the proof of Proposition 7.7,

$$
\begin{aligned}
& d\Bigl[\exp\Bigl(-\int_0^t (c(r,X_r^s)-\delta)dr\Bigr)\bigl(a+\langle \xi_t^s, A_t^s\xi_t^s\rangle^2+\langle \eta_t^s, A_t^s\eta_t^s\rangle\bigr)^{1/2}\Bigr] \\
& \le \frac{1}{2}\exp\Bigl(-\int_0^t (c(r,X_r^s)-\delta)dr\Bigr)\Bigl\{\bigl(a+\langle \xi_t^s, A_t^s\xi_t^s\rangle^2+\langle \eta_t^s, A_t^s\eta_t^s\rangle\bigr)^{-1/2} \\
& \quad \times \bigl[\partial\Delta_t^s - 2(c(t,X_t^s)-\delta)\bigl(a+\langle \xi_t^s, A_t^s\xi_t^s\rangle^2+\langle \eta_t^s, A_t^s\eta_t^s\rangle\bigr)\bigr] \\
& \qquad + C|\xi_t^s|^2 dt + dm_t\Bigr\}.
\end{aligned}
$$

By assumption, $\partial\Delta_t \le (c(t,X_t^s)-\delta)\langle \eta_t^s, A_t^s\eta_t^s\rangle \le 2(c(t,X_t^s)-\delta)\langle \eta_t^s, A_t^s\eta_t^s\rangle$ since c is greater than δ, so that

$$
\begin{aligned}
& d\Bigl[\exp\Bigl(-\int_0^t (c(r,X_r^s)-\delta)dr\Bigr)\bigl(a+\langle \xi_t^s, A_t^s\xi_t^s\rangle^2+\langle \eta_t^s, A_t^s\eta_t^s\rangle\bigr)^{1/2}\Bigr] \\
& \le \exp\Bigl(-\int_0^t (c(r,X_r^s)-\delta)dr\Bigr)\bigl\{C|\xi_t^s|^2 dt + dm_t\bigr\}.
\end{aligned}
$$

Integrating from 0 to $t\wedge\tau_{\mathcal{U}}$, taking the expectation and letting a tend to 0,

$$
\begin{aligned}
& \mathbb{E}\Bigl[\exp\Bigl(-\int_0^{t\wedge\tau_{\mathcal{U}}} \bigl(c(r,X_r^s)-\delta\bigr)dr\Bigr)\bigl(\langle \xi_{t\wedge\tau_{\mathcal{U}}}^s, A_{t\wedge\tau_{\mathcal{U}}}^s\xi_{t\wedge\tau_{\mathcal{U}}}^s\rangle^2 \\
& \qquad + \langle \eta_{t\wedge\tau_{\mathcal{U}}}^s, A_{t\wedge\tau_{\mathcal{U}}}^s\eta_{t\wedge\tau_{\mathcal{U}}}^s\rangle\bigr)^{1/2}\Bigr] \\
& \le \bigl(\langle \gamma'(s), A(\gamma(s))\gamma'(s)\rangle^2 + \langle \gamma''(s), A(\gamma(s))\gamma''(s)\rangle\bigr)^{1/2} \\
& \qquad + C\mathbb{E}\int_0^{t\wedge\tau_{\mathcal{U}}}\Bigl[\exp\Bigl(-\int_0^r (c(u,X_u^s)-\delta)du\Bigr)|\xi_r^s|^2\Bigr]dr.
\end{aligned}
$$

Obviously, the above inequality applies with $\delta/2$ instead of δ. Then, from Proposition 7.7, the last term in the RHS has the form

$$
\begin{aligned}
& \mathbb{E}\int_0^{t\wedge\tau_{\mathcal{U}}}\Bigl[\exp\Bigl(-\int_0^r (c(u,X_u^s)-\delta/2)du\Bigr)|\xi_r^s|^2\Bigr]dr \\
& \le \int_0^{+\infty}\Bigl[\exp\bigl(-(\delta/2)r\bigr)\mathbb{E}\Bigl[\exp\Bigl(-\int_0^{r\wedge\tau_{\mathcal{U}}} (c(u,X_u^s)-\delta)du\Bigr)|\xi_{r\wedge\tau_{\mathcal{U}}}^s|^2\Bigr]dr\Bigr] \\
& \le C\langle \gamma'(s), A(\gamma(s)), \gamma'(s)\rangle \int_0^{+\infty}\exp\bigl(-(\delta/2)r\bigr)dr,
\end{aligned}
$$

for a possibly new value of C, possibly depending on δ as well. This completes the proof of (7.22).

We now investigate w_T''. Following the proof of (7.19), we claim

$$
\begin{aligned}
|w_T''(s)| \le C\mathbb{E}\Bigl[\int_0^T \exp\Bigl(-\int_0^t c(r,X_r^s)dr\Bigr)\bigl[|\eta_t^s| + \int_0^t |\eta_r^s|dr \\
+ |\xi_t^s|^2 + \int_0^t |\xi_r^s|^2 dr + |\xi_t^s|\int_0^t |\xi_r^s|dr + \Bigl(\int_0^t |\xi_r^s|dr\Bigr)^2\bigr]\Bigr].
\end{aligned}
\tag{7.24}
$$

We now apply (7.21) and (7.22). For some possibly new value of the constant C, also depending on the L^∞ norms (on $\mathcal{U}$) of c, $D_x c$, $D^2_{x,x}c$, f, $D_x f$ and $D^2_{x,x}f$,

$$\mathbb{E}\Big[\int_0^T \exp\Big(-\int_0^t c(r, X_r^s)dr\Big)[|\eta_t^s| + \int_0^t |\eta_r^s|dr]\Big] \leq C\big(|\gamma'(s)|^2 + |\gamma''(s)|\big). \tag{7.25}$$

This shows how to deal with the terms in η^s in (7.24). The terms in ξ^s can be handled as follows. Note from Young's inequality and Cauchy-Schwarz inequality that

$$\begin{aligned} &|\xi_t^s|^2 + \int_0^t |\xi_r^s|^2 dr + |\xi_t^s| \int_0^t |\xi_r^s|dr + \Big(\int_0^t |\xi_r^s|dr\Big)^2 \\ &\leq C\Big(|\xi_t^s|^2 + (1+t)\int_0^t |\xi_r^s|^2\Big), \quad t \geq 0. \end{aligned} \tag{7.26}$$

Following (7.21), we complete the proof. □

7.4. **Conclusion.** Before we carry on the analysis of the Monge-Ampère equation, we mention the following points:

(1) We let the reader adapt the statements of Propositions 7.7 and 7.9 to the complex case, then considering A as an Hermitian functional.
(2) As well guessed from Proposition 6.9, the (random) function c in the statements of Propositions 7.7 and 7.9 must be understood as $\mathrm{Trace}(a_t D^2_{z,\bar{z}}\psi(z))$ in the specific framework of Monge-Ampère.
(3) We also emphasize how the rule obtained by Krylov has a very simple form. The whole problem is now to compare two quadratic (or Hermitian in the complex case) forms: $\xi \in \mathbb{R}^d \mapsto \partial\Gamma_t(x,\xi)$ and $\xi \in \mathbb{R}^d \mapsto (c(t,x)-\delta)|\xi|^2$, with $t \geq 0$ and $x \in \mathbb{R}^d$ (or x in a domain of $\mathbb{R}^d$ or $\mathbb{C}^d$: for instance $\mathcal{D}$ in the Monge-Ampère case). If comparison holds, then both the first and second-order derivatives of w_T in the statement of Proposition 7.7 can be controlled uniformly in T. In the Hamilton-Jacobi-Bellman framework, the comparison rule between $\partial\Gamma_t(x,\xi)$ and $(c(t,x)-\delta)|\xi|^2$ must hold for any value of the underlying parameter (denoted by σ in the specific case of Monge-Ampère, see Proposition 6.9). Obviously, establishing such a comparison rule might be really challenging in practice: it is indeed in the Monge-Ampère case!
(4) Below, we sometimes call the process $(\partial\Gamma_t^s)_{t\geq 0}$ in Definition 7.6 *derivative quantity* itself whereas the *derivative quantity* stands for the the process $(\langle \xi_t^s, A(X_t^s)\xi_t^s\rangle)_{t\geq 0}$. We feel that it is not confusive for the reader.

8. Almost Proof of the $\mathcal{C}^1$ Regularity

In this section, we explain how to derive the $\mathcal{C}^1$ property of the solution to Monge-Ampère equation from the program developed in the previous Subsection 7.4. Unfortunately, we are not able to provide a completely rigorous proof at this stage of the notes: some "holes" are indeed left open in the proof. Specifically, some quantities under consideration are not rigorously shown to be differentiable. The plan is thus the following: we here explain how things work without paying too much attention to the differentiability arguments and we postpone to the final Section 9 the complete argument. We will deal with the second-order estimates in Section 9 as well.

For all these reasons, the following statement is called a "Meta-Theorem":

Meta-Theorem 8.1. *Assume that Assumption* (**A**) *is in force and keep the notation of Proposition 6.9. Then, up to the proof of some differentiability properties, it may be shown that, for any compact subset $\mathcal{K} \subset \mathcal{D}$, there exists a constant C, depending on* (**A**) *and $\mathcal{K}$ only, such that, for every smooth curve $\gamma : [-1, 1] \to \mathcal{D}$, the function $s \mapsto V(\gamma(s))$ is Lipschitz with $C\|\gamma'\|_\infty$ as Lipschitz constant.*

Obviously, the whole idea is to apply Points (2) and (3) in Conclusion 7.4 to the solution of the rescaled SDE (6.5), i.e.

$$dZ_t^s = \psi^{1/2}(Z_t^s)\sigma_t dB_t + a_t D_{\bar{z}}^* \psi(Z_t^s)dt, \quad t \geq 0. \tag{8.1}$$

with $Z_0^s = \gamma(s)$, where $\gamma : s \in [-1, 1] \mapsto \gamma(s) \in \mathcal{D}$ is a curve as in the statement of Theorem 8.1. (Note that the compact set $\mathcal{K}$ is not specified at this stage of the proof.) Here, $(\sigma_t)_{t\geq 0}$ denotes a generic control process (i.e. a progressively-measurable process with values in $\mathbb{C}^{d\times d}$ such that $\mathrm{Trace}(\sigma\bar{\sigma}_t^*) = 1$.)

The reader may then easily understand what "Meta" means: because of the exponent 1/2, the function $\psi^{1/2}$ is singular at the boundary so that Theorems 7.2 and 7.4 do not apply to Eq. (8.1). In particular, it may be a bit tricky to establish the differentiability of $(Z_t^s)_{t\geq 0}$ w.r.t. s. As announced above, we forget this difficulty in the whole section and assume that Eq. (8.1) is differentiable in the mean w.r.t. s. Setting $\zeta_t^s = dZ_t^s/ds$, $t \geq 0$, we write (at least formally)

$$\begin{aligned} d\zeta_t^s &= \psi^{-1/2}(Z_t^s)\mathrm{Re}\big[D_z\psi(Z_t^s)\zeta_t^s\big]\sigma_t dB_t \\ &\quad + \big[a_t D_{\bar{z},z}\psi(Z_t^s)\zeta_t^s + a_t D_{\bar{z},\bar{z}}\psi(Z_t^s)\bar{\zeta}_t^s\big]dt. \end{aligned} \tag{8.2}$$

Applying Itô's formula, we could compute the dynamics of $(|\zeta_t^s|^2)_{t\geq 0}$ as in (7.16) and thus express the form of the associated derivative quantity. We won't do it here: the strategy fails when applied in a straightforward way. Said differently, there are very little chances to be able to bound the *derivative quantity* as in the statements of Propositions 7.7 and 7.9.

8.1. **Procedure to Estimate the *Derivative Quantity* in the General Case.**

The major idea of Krylov consists in *perturbing* as most as possible the probabilistic ingredients of the Monge-Ampère equation to improve the long-run control of the *derivative quantity*. Here, the word "perturbing" doesn't mean that we are seeking for another new representation: the general structure given by Proposition 6.9 is the right one. The whole problem is to *perturb* it in a convenient way to obtain the desired long-run estimate.

There are three general ways to perturb the system:

(1) since the problem is stationary, time speed may be changed,
(2) using stochastic processes theory, the underlying probability measure may be perturbed itself,
(3) finally, additional "ghost" control parameters may be plugged into the control representation and used as perturbation parameters.

We here try to explain the main ideas of this *perturbation* procedure. In the next subsections, we will show how to apply them to the Monge-Ampère equation explicitly. Unfortunately, to do so, the method given in Proposition 6.6 must be revisited first.

Having in mind the general notation used in Proposition 6.6, the revisited strategy may be explained as follows. Consider indeed a generic family:

$$w^\beta(s) = \mathbb{E} \int_0^{+\infty} F(\beta_r, X_r^{s,\beta}) dr, \tag{8.3}$$

where

$$dX_t^{s,\beta} = \sigma(\beta_t, X_t^{s,\beta}) dB_t + b(\beta_t, X_t^{s,\beta}) dt, \quad t \geq 0 \; ; \quad X_0^{s,\beta} = \gamma(s),$$

just as in Propositions 6.6 and 6.9. Assume also that, for a given s, we are able to find a family $(\hat{w}^\beta(s+\varepsilon))_\varepsilon$, indexed by a small parameter ε, such that, for any β,

$$\hat{w}^\beta(s+\varepsilon) \leq W(s+\varepsilon) := \sup_\beta w^\beta(s+\varepsilon) \quad \text{and} \quad \hat{w}^\beta(s) = w^\beta(s). \tag{8.4}$$

If the Lipschitz assumption of Proposition 6.6 is satisfied for the family $\hat{w}^\beta(s+\varepsilon)$, i.e.

$$\left| \hat{w}^\beta(s+\varepsilon) - \hat{w}^\beta(s) \right| \leq r_1(\varepsilon), \tag{8.5}$$

(say) for $s, s+\varepsilon \in (-1,1)$ and some function r_1, then

$$W(s+\varepsilon) - w^\beta(s) \geq -r_1(\varepsilon),$$

by the inequality in (8.4), so that

$$W(s+\varepsilon) - W(s) \geq -r_1(\varepsilon), \tag{8.6}$$

by using the equality in (8.4) and by taking the infimum with respect to β. Obviously, if the argument holds for any s in $(-1,1)$, s and $s+\varepsilon$ may be exchanged to bound the increment from above.

Similarly, if the convexity assumption of Proposition 6.6 is satisfied for the family $\hat{w}^\beta(s+\varepsilon)$, i.e.

$$\varepsilon \mapsto \hat{w}^\beta(s+\varepsilon) + r_2(s+\varepsilon) \tag{8.7}$$

is convex (say) for $s, s+\varepsilon, s-\varepsilon \in (-1,1)$ and some function r_2, then, for all β,

$$\begin{aligned} &\liminf_{\varepsilon \to 0} \varepsilon^{-2} \big(W(s+\varepsilon) + r_2(s+\varepsilon) + W(s-\varepsilon) + r_2(s-\varepsilon) \\ &\qquad - 2W(s) - 2r_2(s) \big) \\ &\geq \liminf_{\varepsilon \to 0} \varepsilon^{-2} \big(\hat{w}^\beta(s+\varepsilon) + r_2(s+\varepsilon) + \hat{w}^\beta(s-\varepsilon) + r_2(s-\varepsilon) \\ &\qquad - 2W(s) - 2r_2(s) \big). \end{aligned}$$

Choosing β of the form β^ε so that

$$w^{\beta^\varepsilon}(s) \geq W(s) - \varepsilon^3,$$

we obtain

$$\begin{aligned} &\liminf_{\varepsilon \to 0} \varepsilon^{-2} \big(W(s+\varepsilon) + r_2(s+\varepsilon) + W(s-\varepsilon) + r_2(s-\varepsilon) \\ &\qquad - 2W(s) - 2r_2(s) \big) \\ &\geq \liminf_{\varepsilon \to 0} \varepsilon^{-2} \big(\hat{w}^{\beta^\varepsilon}(s+\varepsilon) + r_2(s+\varepsilon) + \hat{w}^{\beta^\varepsilon}(s-\varepsilon) + r_2(s-\varepsilon) \\ &\qquad - 2\hat{w}^{\beta^\varepsilon}(s) - 2r_2(s) \big). \end{aligned} \tag{8.8}$$

Now, by convexity, the right-hand side is non-negative. (Pay attention, we say so without passing to the limit.) If such a strategy holds for all s in $(-1,1)$, we deduce that $W+r_2$ is convex.

8.2. **Enlarging the Set of Controls.** We now explain how the family $(\hat{w}^{\beta})_{\beta>0}$ can be constructed in the framework of Monge-Ampère.

The starting point is the following: in the specific case of Hamilton-Jacobi-Bellman equations, the set of controls may exhibit some invariance properties; if so, it is conceivable to perturb the system along some transformation that let the system invariant. For instance, for the Monge-Ampère equation, the generic matricial control $(\sigma_t)_{t\geq 0}$ can be replaced by $(\exp(p_t)\sigma_t)_{t\geq 0}$ for some process $(p_t)_{t\geq 0}$ with values in the set of anti-Hermitian matrices: obviously, the trace of $\exp(p_t)a_t\exp(\bar{p}_t^*)=\exp(p_t)a_t\exp(-p_t)$ is still equal to 1.

The auxiliary control parameter $(p_t)_{t\geq 0}$ appears as a "ghost" parameter along which the system may be perturbed. To explain how things work, we go back to Eq. (8.1):

$$dZ_t^s = \psi^{1/2}(Z_t^s)\sigma_t dB_t + a_t D_{\bar{z}}^*\psi(Z_t^s)dt, \quad t\geq 0, \tag{8.9}$$

which is the generic controlled equation used to represent the Monge-Ampère equation as the value function of some optimization problem with an infinite horizon.

As said in introduction of Section 8, we may consider a curve $(\gamma(s))_{s\in[-1,1]}$ with values in $\mathcal{D}$. For a fixed value of s, we define $(\hat{Z}_t^s)_{t\geq 0}$ as above: it is the solution of Eq. (8.9) (or equivalently of Eq. (8.1)) with $\hat{Z}_0^s=\gamma(s)$ as initial solution, so that $\hat{Z}_t^s=Z_t^s$ for any $t\geq 0$. Now, for ε in the neighborhood of 0 (but different from 0), we define $(\hat{Z}_t^{s+\varepsilon})_{t\geq 0}$ as the solution of

$$\begin{aligned} &d\hat{Z}_t^{s+\varepsilon} \\ &= \psi^{1/2}(\hat{Z}_t^{s+\varepsilon})\exp\bigl(P(\hat{Z}_t^s,\hat{Z}_t^{s+\varepsilon}-\hat{Z}_t^s)\bigr)\sigma_t dB_t \\ &\quad + \exp\bigl(P(\hat{Z}_t^s,\hat{Z}_t^{s+\varepsilon}-\hat{Z}_t^s)\bigr)a_t\exp\bigl(\bar{P}^*(\hat{Z}_t^s,\hat{Z}_t^{s+\varepsilon}-\hat{Z}_t^s)\bigr)D_{\bar{z}}^*\psi(\hat{Z}_t^{s+\varepsilon})dt, \end{aligned} \tag{8.10}$$

$t\geq 0$, with $\hat{Z}_0^{s+\varepsilon}=\gamma(s+\varepsilon)$ as initial condition. Here $P(z,z')$ is some function of the parameters z in $\mathcal{D}$ and z' in $\mathbb{C}^d$ with values in the set of anti-Hermitian matrices. It is assumed to be regular in z', with bounded derivatives, uniformly in z so that existence and uniqueness hold for (8.10). (See the proof of Proposition 6.7.) It is also assumed to satisfy $P(z,0)=0$ so that $(\hat{Z}_t^{s+\varepsilon})_{t\geq 0}$ matches $(Z_t^s)_{t\geq 0}$ in (8.9) when $\varepsilon=0$.

The typical choice we perform below for $P(z,z')$ is (at least for z close to the boundary so that $D_z\psi(z)$ is non-zero)

$$\begin{aligned} P(z,z') = \rho\bigl(|D_z\psi(z)|^{-2}\bigl[&D^2_{\bar{z},z}\psi(z)z'D_z\psi(z)+D^2_{\bar{z},\bar{z}}\psi(z)\bar{z}'D_z\psi(z) \\ &- D_{\bar{z}}^*\psi(z)(D^2_{z,\bar{z}}\psi(z)\bar{z}')^* - D_{\bar{z}}^*\psi(z)(D^2_{z,z}\psi(z)z')^*\bigr]\bigr), \end{aligned} \tag{8.11}$$

where ρ is some smooth function from $\mathbb{C}^{d\times d}$ into itself, with compact support, matching the identity on the neighborhood of 0 and preserving the anti-Hermitian structure[7]. (Have in

[7]Think of

$$\rho : (z_{i,j})_{1\leq i,j\leq d}\in\mathbb{C}^{d\times d}\mapsto \rho_1\bigl(\sum_{i,j=1}^d |z_{i,j}|^2\bigr)(z_{i,j})_{1\leq i,j\leq d},$$

where ρ_1 stands for a smooth function from $\mathbb{R}$ to $\mathbb{R}$ with a compact support matching 1 in the neighborhood of zero.

mind that $D_z\psi(z)$ above is seen as a row vector and z' as a column vector.) We let the reader check that $P(z,z')$ is anti-Hermitian.

For ε as above, we set $p_t^{s+\varepsilon} = P(\hat{Z}_t^s, \hat{Z}_t^{s+\varepsilon} - \hat{Z}_t^s) = P(Z_t^s, \hat{Z}_t^{s+\varepsilon} - Z_t^s)$, so that (8.10) has the form

$$d\hat{Z}_t^{s+\varepsilon} = \psi^{1/2}(\hat{Z}_t^{s+\varepsilon}) \exp(p_t^{s+\varepsilon})\sigma_t dB_t + \exp(p_t^{s+\varepsilon}) a_t \exp(-p_t^{s+\varepsilon}) D_{\bar{z}}^* \psi(\hat{Z}_t^{s+\varepsilon}) dt,$$

$t \geq 0$. Now, we can follow Proposition 6.9 and consider

$$\begin{aligned} &\hat{V}^\sigma(s+\varepsilon) \\ &= \mathbb{E}\int_0^{+\infty} \biggl[\exp\biggl(\int_0^t \mathrm{Trace}[\exp(p_r^{s+\varepsilon}) a_r \exp(-p_r^{s+\varepsilon}) D_{z,\bar{z}}^2 \psi(\hat{Z}_r^{s+\varepsilon})] dr\biggr) \\ &\qquad \times F\bigl(\det(a_t), \exp(p_t^{s+\varepsilon}) a_t \exp(-p_t^{s+\varepsilon}), \hat{Z}_t^{s+\varepsilon}\bigr)\biggr] dt. \end{aligned} \tag{8.12}$$

(Pay attention that the determinant of a_t is the same as the determinant of the perturbed matrix $\exp(p_t^{s+\varepsilon}) a_t \exp(-p_t^{s+\varepsilon})$.) Clearly, we have $\hat{V}^\sigma(s) = V^\sigma(\gamma(s))$ (see the notation of Proposition 6.9). Moreover, $\hat{V}^\sigma(s+\varepsilon) \leq \sup_\sigma(V^\sigma(\gamma(s+\varepsilon)))$. (The control $(\exp(p_t^{s+\varepsilon})\sigma_t)_{t\geq 0}$ is a particular control of the same type as $(\sigma_t)_{t\geq 0}$.)

Differentiating (8.12) with respect to ε, we expect[8] a generic expression of the form

$$\begin{aligned} &\frac{d}{d\varepsilon}\bigl[\hat{V}^\sigma(s+\varepsilon)\bigr]_{|\varepsilon=0} \\ &= \mathbb{E}\int_0^{+\infty} \biggl[\exp\biggl(\int_0^t \mathrm{Trace}[a_r D_{z,\bar{z}}^2 \psi(Z_r^s)] dr\biggr) \\ &\quad \times \biggl\{ \Lambda_t^{1,s}\pi_t^s + \bar{\Lambda}_t^{1,s}\bar{\pi}_t^s + \Lambda_t^{2,s}\hat{\zeta}_t^s + \bar{\Lambda}_t^{2,s}\bar{\hat{\zeta}}_t^s \\ &\qquad + \int_0^t \bigl(\Lambda_r^{3,s}\pi_r^s + \bar{\Lambda}_r^{3,s}\bar{\pi}_r^s + \Lambda_r^{4,s}\hat{\zeta}_r^s + \bar{\Lambda}_r^{4,s}\bar{\hat{\zeta}}_r^s\bigr) dr \biggr\} dt\biggr]. \end{aligned} \tag{8.13}$$

Here, $\Lambda_r^{i,s}, \bar{\Lambda}_r^{i,s}$, $i=1,2$, stand for the derivatives of the coefficients appearing in (8.12) and

$$\hat{\zeta}_t^s = \frac{d}{d\varepsilon}\bigl[\hat{Z}_t^{s+\varepsilon}\bigr]_{|\varepsilon=0} \text{ and } \pi_t^s = \frac{d}{d\varepsilon}\bigl[p_t^{s+\varepsilon}\bigr]_{|\varepsilon=0}.$$

Since $p_t^{s+\varepsilon} = P(\hat{Z}_t^s, \hat{Z}_t^{s+\varepsilon} - \hat{Z}_t^s)$, the term π_t^s writes as $D_{z'}P(\hat{Z}_t^s, 0)\hat{\zeta}_t^s + D_{\bar{z}'}P(\hat{Z}_t^s, 0)\bar{\hat{\zeta}}_t^s$ so that (8.13) reduces to

$$\begin{aligned} &\frac{d}{d\varepsilon}\bigl[\hat{V}^\sigma(s+\varepsilon)\bigr]_{|\varepsilon=0} \\ &= \mathbb{E}\int_0^{+\infty} \biggl[\exp\biggl(\int_0^t \mathrm{Trace}[a_r D_{z,\bar{z}}^2 \psi(Z_r^s)] dr\biggr) \\ &\quad \times \biggl\{ \hat{\Lambda}_t^{1,s}\hat{\zeta}_t^s + \bar{\hat{\Lambda}}_t^{1,s}\bar{\hat{\zeta}}_t^s + \int_0^t \bigl(\hat{\Lambda}_r^{2,s}\hat{\zeta}_r^s + \bar{\hat{\Lambda}}_r^{2,s}\bar{\hat{\zeta}}_r^s\bigr) dr \biggr\}\biggr] dt, \end{aligned} \tag{8.14}$$

for two new coefficients $\hat{\Lambda}^{1,s}$ and $\hat{\Lambda}^{2,s}$.

[8] We here say "expect" only since the differentiation argument under the integral symbol is not justified at this stage of the proof.

Before we carry on the analysis, we emphasize that the rigorous proof of (8.14) is far from being easy: it relies on a differentiation argument under the integral symbol that may be very difficult to justify because of the long-run integration. To overcome this problem, a possible strategy is to multiply F by some smooth cut-off function $\phi(\cdot/S)$, S standing for a large positive real and ϕ for a function matching 1 on some $[0,1]$ and vanishing on $[2,+\infty)$. In that case, the differentiation is expected to make sense: for example, it makes sense in the framework of Definition 7.3 because of the supremum over t in $[0,T]$ in the differentiability property. Obviously, the infinite horizon framework can be recovered by letting S tend to $+\infty$ at the end of the analysis, provided the bound we have for the RHS in (8.14) is uniform in the cut-off procedure[9].

The basic argument to bound the RHS in (8.14) is the following. By the very assumption on the coefficients and for the typical choice of P we have in mind, the terms $\hat{\Lambda}^{1,s}$ and $\hat{\Lambda}^{2,s}$ are bounded in the neighborhood of the boundary only, i.e. for $\hat{Z}_t^s = Z_t^s$ close to $\partial\mathcal{D}$. (Indeed, have in mind that $D_z\psi$ is non-zero in the neighborhood of $\partial\mathcal{D}$.) Just for the moment, assume that they are bounded on the whole time interval $[0,+\infty)$. Then, to bound the right-hand side above, it is sufficient to prove an equivalent of (7.17), i.e.

$$\mathbb{E}\Big[\exp\big(-\int_0^t c_r dr\big)|\hat{\zeta}_t^s|^2\Big] \leq \exp(-\delta t)|\hat{\zeta}_0^s|^2 = \exp(-\delta t)|\gamma'(s)|^2 \tag{8.15}$$

for all $t \geq 0$, with $-c_r = \mathrm{Trace}[a_r D^2_{z,\bar{z}}\psi(Z_r^s)]$.

In some sense, we are reduced to the original problem of long-run estimate for the derivative of the diffusion process, but for a new derivative $\hat{\zeta}^s$, namely for the solution of the SDE

$$\begin{aligned} d\hat{\zeta}_t^s &= \big[D_z[\psi^{1/2}](Z_t^s) + \psi^{1/2}(Z_t^s)D_{z'}P(Z_t^s,0)\big]\hat{\zeta}_t^s\sigma_t dB_t \\ &\quad + \big[D_{\bar{z}}[\psi^{1/2}](Z_t^s) + \psi^{1/2}(Z_t^s)D_{\bar{z}'}P(Z_t^s,0)\big]\bar{\hat{\zeta}}_t^s\sigma_t dB_t \\ &\quad + \big\{\big(D_{z'}P(Z_t^s,0)\hat{\zeta}_t^s + D_{\bar{z}'}P(Z_t^s,0)\bar{\hat{\zeta}}_t^s\big)a_t \\ &\qquad - a_t\big(D_{z'}P(Z_t^s,0)\hat{\zeta}_t^s + D_{\bar{z}'}P(Z_t^s,0)\bar{\hat{\zeta}}_t^s\big)\big\}D_{\bar{z}}^*\psi(Z_t^s)dt \\ &\quad + a_t\big[D^*_{\bar{z},z}\psi(Z_t^s)\hat{\zeta}_t^s + D^*_{\bar{z},\bar{z}}\psi(Z_t^s)\bar{\hat{\zeta}}_t^s\big]dt, \quad t \geq 0, \end{aligned} \tag{8.16}$$

with the initial condition $\hat{\zeta}_0^s = \gamma'(s)$. The whole point is then to check that the typical choice (8.11) for $P(z,z')$ permits to derive the long-run estimate (8.15). Unfortunately, we will see below that it permits to obtain (8.15) for Z_t^s close to $\partial\mathcal{D}$ only. (Actually, this is well-guessed: remember that, for the typical choice we have in mind for $P(z,z')$, we cannot bound $\hat{\Lambda}^{1,s}$ and $\hat{\Lambda}^{2,s}$ away from the boundary. Indeed, $P(z,z')$ may explode for z away from the boundary.)

The strategy we follow below consists in localizing the perturbation argument. If the starting point $\gamma(s)$ of Z^s is close enough to the boundary, the perturbation argument applies up to the stopping time $\mathfrak{t} := \inf\{t \geq 0 : \psi(Z_t^s) \geq \epsilon\}$, ϵ standing for some small positive parameter[10]; if the starting point $\gamma(s)$ of Z^s is far away from the boundary, we can apply the perturbation argument when $(\psi(Z_t^s))_{t\geq 0}$ becomes small enough, i.e. when $(Z_t^s)_{t\geq 0}$ enters into the neighborhood of $\partial\mathcal{D}$. Specifically, if $\mathfrak{s}$ is some (finite) stopping time at which $\psi(Z_{\mathfrak{s}}^s) < \epsilon$, we can apply the perturbation argument up to the stopping time $\mathfrak{t} := \inf\{t \geq \mathfrak{s} : \psi(Z_t^s) \geq \epsilon\}$:

[9]We will detail this argument in Section 9 rigorously.

[10]Pay attention that ϵ and ε stand for two different parameters.

Proposition 8.2. *Let $S > 0$ be a positive real, ϕ be a smooth function from $\mathbb{R}_+$ to $[0,1]$ matching 1 on $[0,1]$ and 0 outside $[0,2]$, $\epsilon > 0$ be a small enough real such that $|D_z\psi(z)| > 0$ for $\psi(z) \le \epsilon$ and $\mathfrak{s}$ be some (finite) stopping time such that $\psi(Z^s_{\mathfrak{s}}) < \epsilon$. For $\mathfrak{t} := \inf\{t \ge \mathfrak{s} : \psi(Z^s_t) \ge \epsilon\}$, consider some process $(\hat{Z}^{s+\varepsilon}_t)_{0\le t\le \mathfrak{t}}$ for which $([d/d\varepsilon](\hat{Z}^{s+\varepsilon}_t)_{|\varepsilon=0})_{0\le t\le\mathfrak{t}}$ and $([d^2/d\varepsilon^2](\hat{Z}^{s+\varepsilon}_t)_{|\varepsilon=0})_{0\le t\le\mathfrak{t}}$ exist and for which the perturbed SDE (8.10) holds from $\mathfrak{s}$ to $\mathfrak{t}$ and define*

$$
\begin{aligned}
&\hat{V}^{\sigma,\mathfrak{s},\mathfrak{t}}_S(s+\varepsilon)\\
&= \mathbb{E}\int_{\mathfrak{s}}^{\mathfrak{t}} \Bigl[\exp\Bigl(\int_0^t \mathrm{Trace}[\exp(p^{s+\varepsilon}_r)a_r\exp(-p^{s+\varepsilon}_r)D^2_{z,\bar{z}}\psi(\hat{Z}^{s+\varepsilon}_r)]dr\Bigr)\\
&\qquad \times F\bigl(\det(a_t),\exp(p^{s+\varepsilon}_t)a_t\exp(-p^{s+\varepsilon}_t),\hat{Z}^{s+\varepsilon}_t\bigr)\phi\bigl(\frac{t}{S}\bigr)\Bigr]dt,
\end{aligned}
\tag{8.17}
$$

as the cut-off localized version of (8.12)*, with $p^{s+\varepsilon}_t = P(Z^s_t, \hat{Z}^{s+\varepsilon}_t - Z^s_t)$, $\mathfrak{s} \le t \le \mathfrak{t}$, P being given by* (8.11)*.*

If the differentiation operator w.r.t. ε and the expectation and integral symbols in the RHS of (8.17) *can be exchanged, then there exists a constant $C > 0$, depending on Assumption (**A**) and on ϵ only (in particular, it is independent of S and $(\sigma_t)_{t\ge0}$), such that*

$$
\begin{aligned}
&\Bigl|\frac{d}{d\varepsilon}\bigl[\hat{V}^{\sigma,\mathfrak{s},\mathfrak{t}}_S(s+\varepsilon)\bigr]_{|\varepsilon=0}\Bigr|\\
&\quad \le C\mathbb{E}\Bigl[\int_{\mathfrak{s}}^{\mathfrak{t}} \exp\Bigl(\int_0^t \mathrm{Trace}[a_r D^2_{z,\bar{z}}(Z^s_t)]dr\Bigr)\bigl[|\hat{\zeta}^s_t| + \int_0^t |\hat{\zeta}^s_r|dr\bigr]dt\Bigr],
\end{aligned}
$$

where $\hat{\zeta}^s_t = [d/d\varepsilon](\hat{Z}^{s+\varepsilon}_t)_{|\varepsilon=0}$.

Say a word about the concrete meaning of Proposition 8.2: from time 0 to time $\mathfrak{s}$, the process $(\hat{Z}^{s+\varepsilon}_t)_{0\le t\le\mathfrak{s}}$ is chosen abritrarily provided it be twice differentiable (in the mean) w.r.t. ε. Below, we explicitly say how it is chosen: roughly speaking, it is built from another (local) perturbation argument. We also emphasize, that the value function $\hat{V}^{\sigma,\mathfrak{s},\mathfrak{t}}_S$ has no straightforward connection with the original V: again, we will see below how to gather all the local value functions into a single one, directly connected to Monge-Ampère.

Obviously, we can iterate the argument to bound the second-order derivatives:

Proposition 8.3. *Keep the assumption and notation of Proposition 8.2 and assume that the second-order differentiation operator w.r.t. ε and the expectation and integral symbols in the RHS of* (8.17) *can be exchanged, then there exists a constant $C > 0$, depending on Assumption (**A**) and on ϵ only, such that*

$$
\begin{aligned}
&\Bigl|\frac{d^2}{d\varepsilon^2}\bigl[\hat{V}^{\sigma,\mathfrak{s},\mathfrak{t}}_S(s+\varepsilon)\bigr]\Bigr|\\
&\le C\mathbb{E}\Bigl[\int_{\mathfrak{s}}^{\mathfrak{t}} \exp\Bigl(\int_0^t \mathrm{Trace}[a_r D^2_{z,\bar{z}}(Z^s_t)]dr\Bigr)\\
&\quad \times \Bigl[|\hat{\eta}^s_t| + |\hat{\zeta}^s_t|^2 + \int_0^t |\hat{\eta}^s_r|dr + \int_0^t |\hat{\zeta}^s_r|^2 dr + \Bigl(\int_0^t |\hat{\zeta}^s_r|dr\Bigr)^2\Bigr]dt\Bigr],
\end{aligned}
$$

where $\hat{\eta}^s_t = [d^2/d\varepsilon^2](\hat{Z}^{s+\varepsilon}_t)_{|\varepsilon=0}$.

8.3. **Time Change.** Here is another example of perturbation. The starting point is the following. In the Hamilton-Jacobi-Bellman formulation (5.13) of Monge-Ampère, the normalizing condition for the trace of the matrix a is purely arbitrary. Indeed, the equation remains unchanged when multiplied by any positive constant, so that the trace may be asked to match any other positive real value.

Intuitively, this means that, in Eq. (8.2), the normalizing condition for the trace of $(a_t)_{t\geq 0}$ might be useless, or said differently, that we might consider a rescaled version of $(a_t)_{t\geq 0}$ instead of $(a_t)_{t\geq 0}$ itself.

Now, have in mind that we are here seeking for a perturbed writing of Eq. (8.2) when initialized at $\gamma(s+\varepsilon)$ for ε in the neighborhood of zero. We are thus thinking of rescaling $(a_t)_{t\geq 0}$ by some positive scale function $(|\tau_t^\varepsilon|^2)_{t\geq 0}$ depending on the perturbation variable ε. Here, $(\tau_t^\varepsilon)_{t\geq 0}$ stands for an arbitrary progressively-measurable real-valued process that is differentiable with respect to the parameter ε. Specifically, we consider the perturbed SDE

$$d\hat{Z}_t^{s+\varepsilon} = \psi^{1/2}(\hat{Z}_t^{s+\varepsilon})\tau_t^\varepsilon \sigma_t dB_t + |\tau_t^\varepsilon|^2 a_t D_{\bar{z}}^* \psi(Z_t^{s+\varepsilon})dt, \quad t \geq 0. \tag{8.18}$$

with $\hat{Z}_0^{s+\varepsilon}$ as initial condition. (Solvability is proven as in Proposition 6.7.)

Exactly as in the previous subsection, the perturbation we here choose vanishes at $\varepsilon = 0$, i.e. τ_t^ε is chosen as $T(Z_t^s, \hat{Z}_t^{s+\varepsilon} - Z_t^s)$ for a smooth function $T : (z, z') \in \mathcal{D} \times \mathbb{C}^d \to \mathbb{R}$ such that $T(z,0) = 1$. In other words, $\hat{Z}^s$ and Z^s stand for the same process. In particular, when differentiating $T(Z_t^s, \hat{Z}_t^{s+\varepsilon} - Z_t^s)$ with respect to 0, we obtain $2\mathrm{Re}[D_{z'}T(Z_t^s, 0)\hat{\zeta}_t^s]$ where $\hat{\zeta}_t^s$ stands for the derivative of $Z_t^{s+\varepsilon}$ with respect to ε at $\varepsilon = 0$, i.e.

$$\hat{\zeta}_t^s := \frac{d}{d\varepsilon}\big[\hat{Z}_t^{s+\varepsilon}\big]_{|\varepsilon=0}.$$

The typical choice we have in mind for $T(z,z')$ is

$$T(z,z') = 1 + \rho\big(\psi^{-1}(z)\mathrm{Re}\big[D_z\psi(z)z'\big]\big), \tag{8.19}$$

where ρ is some smooth function with values in $[-1/2, 1/2]$, such that $\rho(0) = 0$ and $\rho'(0) = 1$, so that

$$\mathrm{Re}\big[D_{z'}T(z,0)\zeta\big] = \psi^{-1}(z)\mathrm{Re}\big[D_z\psi(z)\zeta\big], \quad \zeta \in \mathbb{C}^n,$$

and

$$\frac{d}{d\varepsilon}\big[T^2\big(Z_t^s, \hat{Z}_t^{s+\varepsilon} - Z_t^s\big)\big]_{|\varepsilon=0} = 2\psi^{-1}(Z_t^s)\mathrm{Re}\big[D_z\psi(Z_t^s)\hat{\zeta}_t^s\big]. \tag{8.20}$$

The resulting dynamics for $(\hat{\zeta}_t^s)_{t\geq 0}$ is computed below.

The problem is to understand first how this perturbed process is connected with the representation of the solution of Monge-Ampère. Here is the whole point: the process $(\hat{Z}_t^{s+\varepsilon})_{t\geq 0}$ appears as a time-change solution of a SDE of the same type as (8.2). Said in a non-rigorous way, we may think of $(\hat{Z}_t^{s+\varepsilon})$ as $(Z_{\mathfrak{T}_t^\varepsilon}^{s+\varepsilon})_{t\geq 0}$ where $\dot{\mathfrak{T}}_t^\varepsilon = |\tau_t^\varepsilon|^2$, $t \geq 0$, and

$$dZ_t^{s+\varepsilon} = \psi^{1/2}(Z_t^{s+\varepsilon})\frac{\tau^\varepsilon_{(\mathfrak{T}^\varepsilon)_t^{-1}}}{|\tau^\varepsilon_{(\mathfrak{T}^\varepsilon)_t^{-1}}|}\sigma_{(\mathfrak{T}^\varepsilon)_t^{-1}}dB_t + a_{(\mathfrak{T}^\varepsilon)_t^{-1}}D_{\bar{z}}^*\psi(Z_t^{s+\varepsilon})dt, \quad t \geq 0. \tag{8.21}$$

(Here, $(\mathfrak{T}^\varepsilon)^{-1}$ stands for the converse of $\mathfrak{T}^\varepsilon$. We will explain right below why we keep the same notation for this $Z^{s+\varepsilon}$ as in the originial Eq. (8.1).) We won't provide a rigorous proof for this time-change formula[11], but the idea is very intuitive: roughly speaking, the action of the time-change on the dB_t term must be understood as a multiplication by $[\dot{\mathfrak{T}}_t^{s+\varepsilon}]^{1/2}$ since dB_t is understood itself as $[dt]^{1/2}$; obviously, the action of the time-change on the dt terms is the same as in an ODE.

Actually, Eq. (8.21) is false. The reader might guess that, one way or another, the time-change affects the dynamics of the Brownian motion $(B_t)_{t\geq 0}$. The right version is

$$dZ_t^{s+\varepsilon} = \psi^{1/2}(Z_t^{s+\varepsilon}) \frac{\tau^{\varepsilon}_{(\mathfrak{T}^{\varepsilon})_t^{-1}}}{|\tau^{\varepsilon}_{(\mathfrak{T}^{\varepsilon})_t^{-1}}|} \sigma_{(\mathfrak{T}^{\varepsilon})_t^{-1}} d\hat{B}_t^{\varepsilon} + a_{(\mathfrak{T}^{\varepsilon})_t^{-1}} D_{\bar{z}}^* \psi(Z_t^{s+\varepsilon}) dt, \quad t \geq 0, \tag{8.22}$$

where

$$\hat{B}_t^{\varepsilon} = \int_0^{(\mathfrak{T}^{\varepsilon})_t^{-1}} |\tau_r^{\varepsilon}| dB_r, \quad t \geq 0.$$

Here, $(\hat{B}_t^{\varepsilon})_{t\geq 0}$ is a Brownian motion again[12] w.r.t. to the time-rescaled filtration $(\mathcal{F}_{(\mathfrak{T}_t^{\varepsilon})^{-1}})_{t\geq 0}$.

Now, the time-rescaled term $((\tau^{\varepsilon}_{(\mathfrak{T}^{\varepsilon})_t^{-1}}/|\tau^{\varepsilon}_{(\mathfrak{T}^{\varepsilon})_t^{-1}}|)\sigma_{(\mathfrak{T}^{\varepsilon})_t^{-1}})_{t\geq 0}$ may be seen as a new control process with $(a_{(\mathfrak{T}^{\varepsilon})_t^{-1}})_{t\geq 0}$ as Hermitian square, so that we are reduced to the original formulation of Monge-Ampère, but w.r.t. to a different Brownian set-up (the set-up is the pair given by the Brownian motion and the underlying filtration). It may be well-understood that the representation of the Monge-Ampère equation is kept preserved by modification of the underlying Brownian set-up[13], so that

$$\begin{aligned} V\big(\gamma(s+\varepsilon)\big) \geq \mathbb{E}\bigg[\int_0^{+\infty} \exp\bigg(\int_0^t \mathrm{Trace}[a_{(\mathfrak{T}^{\varepsilon})_r^{-1}} D^2_{z,\bar{z}} \psi(Z_r^{s+\varepsilon})] dr\bigg) \\ \times F(\det(a_{(\mathfrak{T}^{\varepsilon})_t^{-1}}), a_{(\mathfrak{T}^{\varepsilon})_t^{-1}}, Z_t^{s+\varepsilon}) dt\bigg]. \end{aligned}$$

[11] We refer the reader to the original paper by Krylov [8] for the complete argument.

[12] Clearly, $(\hat{B}_t^{\varepsilon})_{t\geq 0}$ is a martingale with values in $\mathbb{C}^d$. Actually, for any coordinates $1 \leq j,k \leq d$,

$$d[(\hat{B}_t^{\varepsilon})^j (\hat{B}_t^{\varepsilon})^k] = 0, \quad d[(\hat{B}_t^{\varepsilon})^j \overline{(\hat{B}_t^{\varepsilon})^k}] = \delta_{j,k} dt, \tag{8.23}$$

where $\delta_{j,k}$ stands for the Kronecker symbol. There is a famous theorem in stochastic calculus, due to Paul Lévy, that says that any continuous martingale starting from 0 and satisfying (8.23) is a complex Brownian motion of dimension d. Actually, this may be explained as follows: Eq. (8.23), together with the martingale property, provide the local infinitesimal dynamics of $\hat{B}^{\varepsilon}$; this makes the connection between W and the Laplace operator in $\mathbb{R}^{2d}$ through Itô's formula. In some sense, there is one and only one stochastic process associated with the Laplace operator in $\mathbb{R}^{2d}$: the $2d$-dimensional real Brownian motion or, equivalently, the d-dimensional complex Brownian motion. (For further details, we refer the reader to [14, Thm II. 40].)

[13] Actually, the proof is not so easy: the problem is to understand how the modification of the Brownian paths and of the underlying filtration affects the representation. We refer the reader to the monograph by Krylov [4], Remark III.3.10 for a complete discussion.

(Use Proposition 6.9.) Changing time-speed in the integrals above, we deduce that $V(\gamma(s+\varepsilon)) \geq \hat{V}^\sigma(s+\varepsilon)$ where

$$
\begin{aligned}
&\hat{V}^\sigma(s+\varepsilon) \\
&:= \mathbb{E}\Big[\int_0^{+\infty} \exp\Big(\int_0^t \dot{\mathfrak{T}}_r^\varepsilon \mathrm{Trace}[a_r D^2_{z,\bar{z}}\psi(\hat{Z}_r^{s+\varepsilon})]dr\Big) \\
&\qquad\qquad\qquad \times F(\det(a_t), a_t, \hat{Z}_t^{s+\varepsilon})\dot{\mathfrak{T}}_t^\varepsilon dt\Big] \\
&= \mathbb{E}\Big[\int_0^{+\infty} \exp\Big(\int_0^t |\tau_r^\varepsilon|^2 \mathrm{Trace}[a_r D^2_{z,\bar{z}}\psi(\hat{Z}_r^{s+\varepsilon})]dr\Big) \\
&\qquad\qquad\qquad \times F(\det(a_t), a_t, \hat{Z}_t^{s+\varepsilon})|\tau_t^\varepsilon|^2 dt\Big].
\end{aligned}
\tag{8.24}
$$

Of course, when $\varepsilon = 0$, $\hat{V}^\sigma(s) = V^\sigma(s)$ so that $\sup_\sigma[\hat{V}^\sigma(s)] = V(\gamma(s))$.

The reader may notice that everything works as if $(a_t)_{t\geq 0}$ had been multiplied by the scaling factor $(|\tau_t^\varepsilon|^2)_{t\geq 0}$ as discussed at the very beginning of the paragraph: remember indeed that F is homogeneous with respect to a.

It now remains to understand what happens when differentiating (8.24) w.r.t. ε. We let the reader check that the resulting formula for $[d/d\varepsilon](\hat{v}^\sigma(s+\varepsilon))$ is similar to (8.14). Specifically, the terms $\hat{\Lambda}^{1,s}$ and $\hat{\Lambda}^{2,s}$ therein are bounded in the current framework if $D_{z'}T(z,0)$ is bounded. With the typical choice (8.19) we have in mind, it is bounded away from the boundary, i.e. for $\psi(z)$ away from 0. Actually, the main technical problem is the same as in (8.13): the point is to justify the differentiation. To do, we use the same trick as in the previous subsection by considering some cut-off version of F. We thus deduce the analogs of Propositions 8.2 and 8.3:

Proposition 8.4. *Let S be a positive real, ϕ be a smooth function matching one on $[0,1]$ and vanishing outside $[0,2]$, ϵ be a positive real and $\mathfrak{s}$ be some (finite) stopping time such that $\psi(Z^s_{\mathfrak{s}}) > \epsilon$. For $\mathfrak{t} := \inf\{t \geq \mathfrak{s} : \psi(Z^s_t) \leq \epsilon\}$, consider some process $(\hat{Z}_t^{s+\varepsilon})_{0\leq t\leq \mathfrak{t}}$ for which $([d/d\varepsilon](\hat{Z}_t^{s+\varepsilon})_{|\varepsilon=0})_{0\leq t\leq\mathfrak{t}}$ and $([d^2/d\varepsilon^2](\hat{Z}_t^{s+\varepsilon})_{|\varepsilon=0})_{0\leq t\leq\mathfrak{t}}$ exist and for which the perturbed SDE (8.18) holds from $\mathfrak{s}$ to $\mathfrak{t}$ and define*

$$
\begin{aligned}
&\hat{V}_S^{\sigma,\mathfrak{s},\mathfrak{t}}(s+\varepsilon) \\
&= \mathbb{E}\int_{\mathfrak{s}}^{\mathfrak{t}} \Big[\exp\Big(\int_0^t |\tau_r^\varepsilon|^2 \mathrm{Trace}[a_r D^2_{z,\bar{z}}\psi(\hat{Z}_r^{s+\varepsilon})]dr\Big) \\
&\qquad\qquad \times F\big(\det(a_t), a_t, \hat{Z}_t^{s+\varepsilon}\big)\phi\Big(\frac{\mathfrak{T}_t^\varepsilon}{S}\Big)|\tau_t^\varepsilon|^2\Big] dt,
\end{aligned}
$$

as the localized version of (8.24), with $\tau_t^\varepsilon = T(Z_t^s, \hat{Z}_t^{s+\varepsilon} - Z_t^s)$, $\mathfrak{s} \leq t \leq \mathfrak{t}$, T being given by (8.19), and $\dot{\mathfrak{T}}_t^\varepsilon = |\tau_t^\varepsilon|^2$ (with $\mathfrak{T}_0^\varepsilon = 1$).

If the differentiation operators of order 1 and 2 w.r.t. ε and the expectation and integral symbols in the definition of $\hat{V}_S^{\sigma,\mathfrak{s},\mathfrak{t}}$ can be exchanged, there exists a constant $C > 0$, depending *on Assumption* (**A**) *and on* ϵ *only, such that*

$$
\begin{aligned}
\Big|\frac{d}{d\varepsilon}\big[\hat{V}_S^{\sigma,\mathfrak{s},\mathfrak{t}}(s+\varepsilon)\big]\Big| &\leq C\mathbb{E}\bigg[\int_{\mathfrak{s}}^{\mathfrak{t}} \exp\Big(\int_0^t \mathrm{Trace}[a_r D^2_{z,\bar{z}}(Z_t^s)]dr\Big) \\
&\qquad\qquad \times \big[|\hat{\zeta}_t^s| + \int_0^t |\hat{\zeta}_r^s| dr\big] dt\bigg] \\
\Big|\frac{d^2}{d\varepsilon^2}\big[\hat{V}_S^{\sigma,\mathfrak{s},\mathfrak{t}}(s+\varepsilon)\big]\Big| &\leq C\mathbb{E}\bigg[\int_{\mathfrak{s}}^{\mathfrak{t}} \exp\Big(\int_0^t \mathrm{Trace}[a_r D^2_{z,\bar{z}}(Z_t^s)]dr\Big) \\
&\times \bigg[|\hat{\eta}_t^s| + |\hat{\zeta}_t^s|^2 + \int_0^t |\hat{\eta}_r^s| dr + \int_0^t |\hat{\zeta}_r^s|^2 dr + \Big(\int_0^t |\hat{\zeta}_r^s| dr\Big)^2\bigg] dt\bigg],
\end{aligned}
$$

where $\hat{\zeta}_t^s = [d/d\varepsilon](Z_t^{s+\varepsilon})_{|\varepsilon=0}$ *and* $\hat{\eta}_t^s = [d^2/d\varepsilon^2](Z_t^{s+\varepsilon})_{|\varepsilon=0}$.

The reader may wonder about the specific choice for the cut-off. First, the time-change is plugged as an argument of the cut-off function: when performing the change of variable, we recover $(\phi(t/S))_{t\geq 0}$ as cut-off. Second, we emphasize that the cut-off permits to get rid of times t at which $\mathfrak{T}_t^\varepsilon \geq 2S$. By assumption, we know that $|\tau^\varepsilon|^2$ is always greater than $1/4$ so that $\mathfrak{T}_t^\varepsilon$ is always greater than $t/4$, $t \geq 0$. In particular, the cut-off vanishes at times t at which $t/4 \geq 2S$. In other words, the definition of $\hat{V}_S^{\sigma,\mathfrak{s},\mathfrak{t}}$ is understood as a finite horizon value function: this permits to justify the differentation argument w.r.t. ε provided $(\hat{Z}_t^{s+\varepsilon})_{0\leq t\leq \mathfrak{t}}$ satisfies the assumption of Corollary 7.5. (Have in mind that Corollary 7.5 holds in finite horizon.) Unfortunately, because of the singularity of the coefficient $\psi^{1/2}$ in (8.1) in the neighborhood of $\partial\mathcal{D}$, it is not so easy to prove that $(\hat{Z}_t^{s+\varepsilon})_{0\leq t\leq \mathfrak{t}}$ satisfies the assumption of Corollary 7.5. At this stage of the proof, this point is left open: this is the "meta"-part of Meta-Theorem 8.1.

8.4. **Perturbation of the Measure: Girsanov Theorem.** The last perturbation method we here discuss consists in modifying the measure of the underlying probability space. This a typical probabilistic way to estimate the solution of a partial differential equation of second-order: we may refer the reader to the lectures by Krylov in Pisa [9] for a detailed overview; we also mention the personal work [2] and the references therein.

We here explain first how the probability measure may be changed to establish some smoothness property for the solution of a second-order partial differential equation. Generally speaking, the modification of the reference measure is a common argument in stochastic analysis, which turns out to be really efficient to quantify the sensitivity of a system with respect to the input noise. More or less, this is the starting point of the Malliavin Calculus, used to prove by probabilistic tools the so-called "Sum of squares" Theorem due to Hörmander. (See the monograph [13].)

In the specific case of heat equation, the problem may be understood as follows. Indeed, as already explained in (3.1) and (3.2), the solution of the one-dimensional heat equation

$$D_t u(t,x) - \frac{1}{2} D^2_{x,x} u(t,x) = 0, \quad (t,x) \in (0,+\infty) \times \mathbb{R},$$

with an initial condition of the form $u(0,\cdot) = u_0(\cdot)$ (say, with u_0 continuous and bounded) is given by

$$u(t,x) = \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} u_0(y) \exp\big(-\frac{|x-y|^2}{2t}\big) dy, \quad x \in \mathbb{R},$$

Clearly, at fixed $t > 0$, and for any $\varepsilon \in \mathbb{R}$, the Gaussian measures

$$\frac{1}{\sqrt{2\pi t}} \exp\big(-\frac{|x-y|^2}{2t}\big) dy \quad \text{and} \quad \frac{1}{\sqrt{2\pi t}} \exp\big(-\frac{|x+\varepsilon-y|^2}{2t}\big) dy$$

are equivalent, so that $u(t, x+\varepsilon)$ can be written as

$$\begin{aligned} &u(t,x+\varepsilon) \\ &= \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} u_0(y) \exp\big(-\frac{|x+\varepsilon-y|^2 - |x-y|^2}{2t}\big) \exp\big(-\frac{|x-y|^2}{2t}\big) dy \\ &= \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} u_0(x+y) \exp\big(-\frac{|\varepsilon-y|^2 - |y|^2}{2t}\big) \exp\big(-\frac{|y|^2}{2t}\big) dy. \end{aligned}$$

Thinking of the Gaussian density as the density of the (marginal) law of the position of some Brownian B at time t, we may write as well:

$$\begin{aligned} u(t,x+\varepsilon) &= \mathbb{E}\big[u_0(x+B_t) \exp\big(-\frac{|\varepsilon - B_t|^2 - |B_t|^2}{2t}\big)\big] \\ &= \mathbb{E}\big[u_0(x+B_t) \exp\big(\varepsilon \frac{B_t}{t} - \frac{\varepsilon^2}{2t}\big)\big]. \end{aligned}$$

Now, the term $M^\varepsilon = \exp(\varepsilon B_t/t - \varepsilon^2/(2t))$ appears as a density on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which the Brownian motion is defined. Said differently, the representation of $u(t,x+\varepsilon)$ consists in integrating $u_0(x+B_t)$, as for $u(t,x)$, but under the measure $M^\varepsilon \cdot \mathbb{P}$. In particular, the smoothness of $u(t,\cdot)$ with respect to the spatial parameter is directly given by the smoothness of the density M^ε with respect to the parameter ε.

This example is very simple because the change of measure is of finite dimension. Nevertheless, there exists an infinite dimensional counterpart, known as Girsanov Theorem[14].

To understand how things work, go back to the statement of Theorem 7.2 and consider a curve γ of the form $\gamma(s) = x_0 + (T-s)\nu$, where T is some positive real, and x_0 and ν some vectors in $\mathbb{R}^d$. (Recall that, for more simplicity, the framework of Theorem 7.2 is real and not complex.) The whole idea now consists in considering $(X_t^{\gamma(t)})_{0 \le t \le T}$: it both depends on time t through the time index of X and through the initial condition $\gamma(t)$. (Keep in mind that $X_0^{\gamma(t)} = \gamma(t)$.) It can be proven (see e.g. the monograph by Kunita [10]) that

$$dX_t^{\gamma(t)} = b(t, X_t^{\gamma(t)})dt + \sigma(t, X_t^{\gamma(t)})dB_t + \xi_t^{\gamma(t)} dt,$$

where $\xi_t^{\gamma(t)}$ is the value of $\xi_t^s = D_s[X_t^{\gamma(s)}]$ at $s = t$. (That is, $\xi_t^{\gamma(t)} = D_x X_t^{\gamma(t)} \gamma'(t)$. See the statement of Theorem 7.2.)

The big deal is the following. If σ is invertible and σ^{-1} is bounded, uniformly in time and space, we write

$$dX_t^{\gamma(t)} = b(t, X_t^{\gamma(t)})dt + \sigma(t, X_t^{\gamma(t)})\big(dB_t + \sigma^{-1}(t, X_t^{\gamma(t)})\xi_t^{\gamma(t)} dt\big).$$

[14] We won't give the explicit form of Girsanov Theorem here. It would require an additional effort which seems useless. We refer to the monograph by Protter [14].

What Girsanov Theorem says is: we can find a new measure $\mathbb{Q}$, equivalent to $\mathbb{P}$ on the σ-algebra generated by $(B_t)_{0\le t\le T}$, such that the process in parentheses be a Brownian motion, i.e.

$$\left(B_t + \int_0^t \sigma^{-1}(r, X_r^{\gamma(r)})\xi_r^{\gamma(r)} dr\right)_{0\le t\le T},$$

is a Brownian motion under $\mathbb{Q}$[15]. As a consequence, under the new probability measure $\mathbb{Q}$, the process $(X_t^{\gamma(t)})_{0\le t\le T}$ behaves as the initial process $(X_t^{\gamma(0)})_{0\le t\le T}$ under $\mathbb{P}$. In particular, if u stands for the solution of the Cauchy problem

$$D_t u(t,x) + \langle b(t,x), D_x u(t,x)\rangle + \frac{1}{2}\mathrm{Trace}\big[a(t,x)D^2_{x,x}u(t,x)\big] = 0,$$

with the boundary condition $u(T,x) = u_T(x)$. (Note that the problem is set in a backward way for notational simplicity only), the initial condition $u(0,\gamma(0))$ can be written on the same model as (5.3) as $\mathbb{E}_{\mathbb{P}}[u_T(X_T^{\gamma(0)})]$ and therefore as $\mathbb{E}_{\mathbb{Q}}[u_T(X_T^{\gamma(T)})]$. (Here, the indices $\mathbb{P}$ and $\mathbb{Q}$ denote the probability used to perform the integration.) In particular,

$$u(0, x_0 + T\nu) = \mathbb{E}_{\mathbb{Q}}\big[u(T, X_T^{\gamma(T)})\big].$$

Now, the trick is: $\gamma(T) = x_0$ so that

$$u(0, x_0 + T\nu) = \mathbb{E}_{\mathbb{Q}}\big[u(T, X_T^{x_0})\big].$$

Finally, it remains to give the form of $\mathbb{Q}$. It is given by Girsanov Theorem as

$$\begin{aligned}\frac{d\mathbb{Q}}{d\mathbb{P}} &= \rho_T^\nu \\ &:= \exp\Big(-\int_0^T \langle \sigma^{-1}(r, X_r^{\gamma(r)})\xi_t^{\gamma(t)}, dB_t\rangle - \frac{1}{2}\int_0^T |\sigma^{-1}(t, X_t^{\gamma(t)})\xi_t^{\gamma(t)}|^2 dt\Big).\end{aligned}$$

Finally,

$$u(0, x_0 + T\nu) = \mathbb{E}_{\mathbb{P}}\big[u(T, X_T^{x_0})\rho_T^\nu\big].$$

In other words, the regularity of u with respect to the spatial parameter follows from the regularity of ρ_T^ν, independently of the regularity of the boundary condition: this is the typical probabilistic argument to understand the regularizing effect of non-degenerate diffusion operators. Of course, the price to pay is the same as in analysis: the underlying diffusion matrix has to be non-degenerate.

Obviously, this is not the case in the Monge-Ampère problem. However, we will use Girsanov Theorem as a perturbation tool.

The idea is the following: go back to Eq. (8.1) and consider at $s+\varepsilon$ the perturbed dynamics

$$\begin{aligned} d\hat{Z}_t^{s+\varepsilon} &= \psi^{1/2}(\hat{Z}_t^{s+\varepsilon})\sigma_t\big[dB_t + G(Z_t^s, \hat{Z}_t^{s+\varepsilon} - Z_t^s)dt\big] \\ &\quad + a_t D_{\bar{z}}^*\psi(\hat{Z}_t^{s+\varepsilon})dt, \quad t\ge 0. \end{aligned} \tag{8.25}$$

Here, the function G satisfies $G(z,0) = 0$ so that $(\hat{Z}_t^s)_{t\ge 0}$ and $(Z_t^s)_{t\ge 0}$ are equal as required in the perturbation method. When G (seen as a function of two arguments) is a smooth

[15]The reader who knows Girsanov Theorem already may notice that the exponential martingale property should be checked to apply the theorem. Obviously, it should be: actually, the whole argument relies on a localization procedure that is a little bit involved. For simplicity, we do not discuss it here.

function with a compact support, the unique solvability of (8.25) may be proven as in Proposition 6.7: the sketch is given in footnote below [16]. (The reader can skip it.) To make the connection with the original dynamics, we are then seeking for a new measure $\mathbb{P}^{\varepsilon}$ under which the process

$$\Big(\hat{B}_t^{\varepsilon} := B_t + \int_0^t G(Z_r^s, \hat{Z}_r^{s+\varepsilon} - Z_r^s)dr\Big)_{t\geq 0}$$

is a Brownian motion. (So that, under $\mathbb{P}^{\varepsilon}$, the process $(\hat{Z}_t^{s+\varepsilon})_{t\geq 0}$ has the right dynamics.)

What Girsanov Theorem[17] says is the following: if G is bounded, there exists a measure $\mathbb{P}^{\varepsilon}$ given by

$$\begin{aligned}\mathbb{P}^{\varepsilon}(A) = \mathbb{E}\Big[\exp\Big(- \int_0^t 2\mathrm{Re}\big[\langle \bar{G}(Z_r^s, \hat{Z}_r^{s+\varepsilon} - Z_r^s), dB_r\rangle\big] \\ - \int_0^t |G|^2(Z_r^s, \hat{Z}_r^{s+\varepsilon} - Z_r^s)dr\Big)\mathbf{1}_A\Big], \quad A \in \mathcal{F}_t, \quad t \geq 0,\end{aligned} \tag{8.26}$$

under which $(\hat{B}_t^{\varepsilon})_{t\geq 0}$ is a complex Brownian motion of dimension d. (In particular, $\mathbb{P}^{\varepsilon}$ admits a density with respect to $\mathbb{P}$ (and is even equivalent to $\mathbb{P}$) when restricted to the σ-subalgebra $\mathcal{F}_t$, $t \geq 0$.)

We now go back to (8.25): we understand that $(\hat{Z}_t^{s+\varepsilon})_{t\geq 0}$ has the same dynamics as $(Z_t^{s+\varepsilon})_{t\geq 0}$ in (8.1) but with $(B_t)_{t\geq 0}$ replaced by $(\hat{B}_t^{\varepsilon})_{t\geq 0}$. Since $(\hat{B}_t^{\varepsilon})_{t\geq 0}$ is a Brownian motion under $\mathbb{P}^{\varepsilon}$, we expect $(\hat{Z}_t^{s+\varepsilon})_{t\geq 0}$ to have the same dynamics (i.e. the same distribution) under $\mathbb{P}^{\varepsilon}$ as $(Z_t^{s+\varepsilon})_{t\geq 0}$ under $\mathbb{P}$. Under local Cauchy-Lipschitz like type assumption on the coefficients of (8.1), this is true: this is the so-called Yamada and Watanabe Theorem, see e.g. Stroock and Varadhan [17].

---

[16]The argument is almost the same as in Proposition 6.7 but the right martingale to consider in (6.6) is

$$\begin{aligned}m_t = \psi^{-1}(\hat{Z}_t^{s+\varepsilon}) \times \exp\Big(\int_0^t \mathrm{Trace}[a_r D^2_{z,\bar{z}}\psi(Z_r^{s+\varepsilon})]dr \\ - \int_0^t 2\mathrm{Re}\big[\langle \bar{G}(Z_r^s, \hat{Z}_r^{s+\varepsilon} - Z_r^s), dB_r\rangle\big] - \int_0^t |G|^2(Z_r^s, \hat{Z}_r^{s+\varepsilon} - Z_r^s)dr\Big),\end{aligned}$$

$t \geq 0$. Indeed, by Itô's formula, we can prove that it is a local martingale.

Then, denoting by $\tau_n = \inf\{t \geq 0 : \psi^{-1}(Z_t^{s+\varepsilon}) \leq 1/n\}$,

$$\begin{aligned}n^{1/2}\mathbb{P}\{\tau_n \leq t\} &\leq \mathbb{E}\big[\psi^{-1/2}(\hat{Z}_t^{s+\varepsilon})\big] \\ &\leq \mathbb{E}\Big[\psi^{-1}(\hat{Z}_t^{s+\varepsilon})\exp\Big(-\int_0^t 2\mathrm{Re}\big[\langle \bar{G}(Z_r^s, \hat{Z}_r^{s+\varepsilon} - Z_r^s), dB_r\rangle\big]\Big)\Big]^{1/2} \\ &\quad \times \mathbb{E}\Big[\exp\Big(\int_0^t 2\mathrm{Re}\big[\langle \bar{G}(Z_r^s, \hat{Z}_r^{s+\varepsilon} - Z_r^s), dB_r\rangle\big]\Big)\Big]^{1/2} \\ &\leq C\exp(Ct)\mathbb{E}[m_t] = C\exp(Ct)\psi^{-1}(z).\end{aligned}$$

The last line follows from the bound

$$\mathbb{E}\Big[\exp\Big(\int_0^t 2\mathrm{Re}\big[\langle \bar{G}(Z_r^s, \hat{Z}_r^{s+\varepsilon} - Z_r^s), dB_r\rangle\big]\Big)\Big]^{1/2} \leq \exp(C\|G\|_\infty t).$$

See Rogers and Williams [15].

[17]Pay attention that Girsanov Theorem is here given for the complex Brownian motion.

Consider now the perturbed value function

$$
\begin{aligned}
&\hat{V}^{\sigma}(s+\varepsilon) \\
&= \int_0^{+\infty} \mathbb{E}\Big[\exp\Big(-\int_0^t 2\mathrm{Re}\big[\langle \bar{G}(Z_r^s, \hat{Z}_r^{s+\varepsilon} - Z_r^s), dB_r\rangle\big] \\
&\qquad\qquad - \int_0^t |G|^2(Z_r^s, \hat{Z}_r^{s+\varepsilon} - Z_r^s)dr\Big) \\
&\qquad \times \exp\Big(\int_0^t \mathrm{Trace}[a_r D^2_{z,\bar{z}}\psi(\hat{Z}_r^{s+\varepsilon})]dr\Big) F\big(\det(a_t), a_t, \hat{Z}_t^{s+\varepsilon}\big)\Big] dt.
\end{aligned}
\tag{8.27}
$$

(Note that the integral and the expectation have been exchanged in comparison with the original formulation in Proposition 6.9. This new writing permits to apply Girsanov Theorem easily. Nevertheless, by boundedness of F and superharmonicity of ψ, Fubini's Theorem applies and the integrals may be exchanged.) We may write it as

$$
\begin{aligned}
\hat{V}^{\sigma}(s+\varepsilon) = \int_0^{+\infty} \mathbb{E}_{\mathbb{P}^{\varepsilon}}\Big[\exp\Big(\int_0^t \mathrm{Trace}\big[a_r D^2_{z,\bar{z}}\psi(\hat{Z}_r^{s+\varepsilon})\big]dr\Big) \\
\times F\big(\det(a_t), a_t, \hat{Z}_t^{s+\varepsilon}\big)\Big] dt,
\end{aligned}
$$

where $\mathbb{E}_{\mathbb{P}^{\varepsilon}}$ denotes the expectation under $\mathbb{P}^{\varepsilon}$. We then replace $\hat{Z}^{s+\varepsilon}$ by $Z^{s+\varepsilon}$ by saying that the dynamics of the first one under $\mathbb{P}^{\varepsilon}$ are the same as the dynamics of the second one under $\mathbb{P}$. We deduce that the supremum $\sup_{\sigma} \hat{v}^{\sigma}(s+\varepsilon)$ is equal to $V(\gamma(s+\varepsilon))$[18].

It now remains to specify the choice for G. Actually, we can choose it such that

$$
\frac{d}{d\varepsilon}\big[\bar{G}(Z_t^s, \hat{Z}_t^{s+\varepsilon} - Z_t^s)\big]_{|\varepsilon=0} = \Xi(Z_t^s)\hat{\zeta}_t^s,
\tag{8.28}
$$

where $\Xi(z)$ is a complex matrix of size $d\times d$ and $\hat{\zeta}_t^s = [d/d\varepsilon](\hat{Z}_t^{s+\varepsilon})_{|\varepsilon=0}$. (Choose for example $G(z,z') = \Xi(z)\rho(z')$, the function ρ being bounded and satisfying $\rho(0)=0$, $D_{z'}\rho(0) = I_d$ and $D_{\bar{z}'}\rho(0) = 0$.[19]) Below, the matrix $\Xi(z)$ we use is bounded in z on every compact subset of $\mathcal{D}$ only. (In particular, $\Xi(z)$ may explode as z tends to $\partial\mathcal{D}$.)

To complete the argument, it remains to explain what happens when differentiating (8.27) w.r.t. ε. (Again, we assume that we can do so: this is a part of the "meta" in Meta-Theorem 8.1.) The story is a bit different from what we explained above for the two other perturbations. Indeed, when differentiating (8.27), we obtain a new term to bound which is

$$
\mathbb{E}\int_0^{+\infty}\Big|\int_0^t \langle \Xi(Z_r^s)\hat{\zeta}_r^s, dB_r\rangle \exp\Big(\int_0^t \mathrm{Trace}\big[a_r D_{z,\bar{z}}\psi(Z_r^s)\big]dr\Big)\Big| dt.
$$

Here is what we can say:

Lemma 8.5. *Consider a process $(\varsigma_t)_{t\geq 0}$ with values in $\mathbb{C}^d$, solution to a SDE of the form*

$$
d\varsigma_t = \big(\beta_t\varsigma_t + \beta'_t\bar{\varsigma}_t\big)dt + (\alpha_t\varsigma_t + \alpha'_t\bar{\varsigma}_t)dB_t,
$$

[18]Here, the story is the same as for time-change. To have a completely rigorous argument, we should check first that the representation of Monge-Ampère remains the same when the underlying Brownian motion is modified. Again, we refer to Remark III.3.10 in the monograph [4] for a complete discussion.

[19]A typical example is $\rho(z') = (\rho_0(z'_i))_{1\leq i\leq d}$ with $\rho_0(z'_i) = z'_i\exp(-|z'_i|^2)$, $z'_i\in\mathbb{C}$.

the coefficients $(\beta_t)_{t\geq 0}$, $(\beta'_t)_{t\geq 0}$ and $(\alpha_t)_{t\geq 0}$, $(\alpha'_t)_{t\geq 0}$ being $\mathbb{C}^d \otimes \mathbb{C}^d$ and $\mathbb{C}^{d\times d} \otimes \mathbb{C}^d$-valued respectively (i.e. $\beta_t \varsigma_t$ and $\beta'_t \bar{\varsigma}_t$ are in $\mathbb{C}^d$ and $\alpha_t \varsigma_t$ and $\alpha'_t \bar{\varsigma}_t$ are in $\mathbb{C}^{d\times d}$) and being possibly random as well. Set

$$m_t = \int_0^t \langle \Xi_r \varsigma_r, dB_r \rangle, \quad t \geq 0,$$

for another bounded $\mathbb{C}^{d\times d}$-valued process $(\Xi_t)_{t\geq 0}$. Assume finally that $(\Xi_t)_{t\geq 0}$ vanishes when the process $(\psi(Z^s_t))_{t\geq 0}$ is less than some $\epsilon_{00} > 0$. Then, for a non-positive process $(c_t)_{t\geq 0}$,

$$\mathbb{E}\biggl[|m_t| \exp\biggl(\int_0^t c_r dr\biggr)\biggr] \leq C\mathbb{E}\biggl[\int_0^t |\varsigma_r|\bigl(1 + r^{-1/2}\bigr) \exp\biggl(\int_0^r c_u du\biggr) dr\biggr],$$

the constant C only depending on the bound of Ξ and on the bounds of α, α', β and β' at times t for which $\psi(Z^s_t) > \epsilon_{00}/2$.

Proof. We follow the proof of (7.23). We consider a smooth cut-off function φ with values in $[0, 1]$ matching 1 on $[\epsilon_{00}, +\infty)$ and vanishing on $(-\infty, \epsilon_{00}/2]$. Applying Itô's formula, we write

$$\begin{aligned} d\bigl[\varphi(\psi(Z^s_t))\bigr] &= \varphi'(\psi(Z^s_t)) d^{(1)}_t dt + \varphi''(\psi(Z^s_t)) \bigl|d^{(2)}_t\bigr|^2 dt \\ &\quad + \varphi'(\psi(Z^s_t)) \langle d^{(2)}_t, dB_t \rangle + \varphi'(\psi(Z^s_t)) \langle \bar{d}^{(2)}_t, d\bar{B}_t \rangle, \end{aligned}$$

$t \geq 0$, where $(d^{(1)}_t)_{t\geq 0}$ and $(d^{(2)}_t)_{t\geq 0}$ stand for the coefficients of the Itô expansion of $(\psi(Z^s_t))_{t\geq 0}$, i.e.

$$d\bigl[\psi(Z^s_t)\bigr] = d^{(1)}_t dt + \langle d^{(2)}_t, dB_t \rangle + \langle \bar{d}^{(2)}_t, d\bar{B}_t \rangle, \quad t \geq 0.$$

Note also that

$$\begin{aligned} d\bigl[|\varsigma_t|^2\bigr] &= \bigl(2\mathrm{Re}\bigl[\langle \bar{\varsigma}_t, \beta_t \varsigma_t + \beta'_t \bar{\varsigma}_t \rangle\bigr] + |\alpha_t \varsigma_t + \alpha'_t \bar{\varsigma}_t|^2\bigr) dt \\ &\quad + 2\mathrm{Re}\bigl[\langle (\alpha_t \varsigma_t + \alpha'_t \bar{\varsigma}_t)^* \bar{\varsigma}_t, dB_t \rangle\bigr], \quad t \geq 0. \end{aligned}$$

Therefore,

$$\begin{aligned} &d\bigl(|m_t|^2 + t\varphi(\psi(Z^s_t))|\varsigma_t|^2\bigr) \\ &= \bigl[|\Xi_t \varsigma_t|^2 + \varphi(\psi(Z^s_t))|\varsigma_t|^2 \\ &\quad + 2t\varphi(\psi(Z^s_t))\mathrm{Re}\bigl[\langle \bar{\varsigma}_t, \beta_t \varsigma_t + \beta'_t \bar{\varsigma}_t \rangle\bigr] + t\varphi(\psi(Z^s_t))\bigl|\alpha_t \varsigma_t + \alpha'_t \bar{\varsigma}_t\bigr|^2 \\ &\quad + t|\varsigma_t|^2 \varphi'(\psi(Z^s_t)) d^{(1)}_t + t|\varsigma_t|^2 \varphi''(\psi(Z^s_t))\bigl|d^{(2)}_t\bigr|^2 \\ &\quad + 2t\varphi'(\psi(Z^s_t))\mathrm{Re}\bigl[\langle (\alpha_t \varsigma_t + \alpha'_t \bar{\varsigma}_t)^* \bar{\varsigma}_t, \bar{d}^{(2)}_t \rangle\bigr] dt + dn_t, \quad t \geq 0, \end{aligned}$$

where $(n_t)_{t\geq 0}$ stands for a new martingale term whose value may vary from line to line. Then, for any small $a > 0$, by concavity of the function $x \in \mathbb{R}_+ \mapsto (a + x)^{1/2}$ and by the bound $|\Xi_t \varsigma_t|^2 \leq \varepsilon_{00}^{-1/2} |\Xi_t \mathbf{1}_{\{\psi(Z_t^s) \geq \varepsilon_{00}\}}|^2 \varphi^{1/2}(\psi(Z_t^s))|\varsigma_t|^2$,

$$
\begin{aligned}
& d\big(a + |m_t|^2 + t\varphi(\psi(Z_t^s))|\varsigma_t|^2\big)^{1/2} \\
& \leq \frac{1}{2}\big(a + |m_t|^2 + t\varphi(\psi(Z_t^s))|\varsigma_t|^2\big)^{-1/2} \big\{ |\Xi_t \varsigma_t|^2 + \varphi(\psi(Z_t^s))|\varsigma_t|^2 \\
& \quad + 2t\varphi(\psi(Z_t^s))\mathrm{Re}\big[\langle \bar{\varsigma}_t, \beta_t \varsigma_t + \beta_t' \bar{\varsigma}_t \rangle\big] + t\varphi(\psi(Z_t^s))\big|\alpha_t \varsigma_t + \alpha_t' \bar{\varsigma}_t\big|^2 \\
& \quad + t|\varsigma_t|^2 \varphi'(\psi(Z_t^s)) d_t^{(1)} + t|\varsigma_t|^2 \varphi''(\psi(Z_t^s))\big|d_t^{(2)}\big|^2 \\
& \quad + 2t\varphi'(\psi(Z_t^s))\mathrm{Re}\big[\langle (\alpha_t \varsigma_t + \alpha_t' \bar{\varsigma}_t)^* \bar{\varsigma}_t, \bar{d}_t^{(2)} \rangle\big] \big\} dt + dn_t, \\
& \leq C\big(1 + t^{-1/2}\big)|\varsigma_t| dt + dn_t,
\end{aligned}
\tag{8.29}
$$

the constant C here depending on the bound of $(\Xi_t)_{t\geq 0}$, the bounds of the processes $(\alpha_t \mathbf{1}_{\{\psi(Z_t^s) > \epsilon_{00}/2\}})_{t\geq 0}$, $(\alpha_t' \mathbf{1}_{\{\psi(Z_t^s) > \epsilon_{00}/2\}})_{t\geq 0}$, $(\beta_t \mathbf{1}_{\{\psi(Z_t^s) > \epsilon_{00}/2\}})_{t\geq 0}$ and $(\beta_t' \mathbf{1}_{\{\psi(Z_t^s) > \epsilon_{00}/2\}})_{t\geq 0}$ and the supremum norm of $\varphi'/\varphi^{1/2}$ and $\varphi''/\varphi^{1/2}$. (Note that $(d_t^{(1)})_{t\geq 0}$ and $(d_t^{(2)})_{t\geq 0}$ are bounded by known constants.) In particular, C is independent of a.

Now, we can choose φ such that $\varphi'/\varphi^{1/2}$ and $\varphi''/\varphi^{1/2}$ be bounded. For example, think of $\varphi(x) = \exp[-\epsilon_{00}^2/(x^2 - (\epsilon_{00}/2)^2)]$ for $x \in (\epsilon_{00}/2, \epsilon_{00}/\sqrt{2})$, $\varphi(x) = 0$ for $x \leq \epsilon_{00}/2$, $\varphi(x) = 1$ for $x \geq \epsilon_{00}$ and $\varphi(x) \in [\exp(-4), 1]$ for $x \in (\epsilon_{00}/\sqrt{2}, \epsilon_{00})$. As a consequence, we can assume that the constant C in (8.29) only depends on the bounds of $(\Xi_t)_{t\geq 0}$, $(\alpha_t \mathbf{1}_{\{\psi(Z_t^s) > \epsilon_{00}/2\}})_{t\geq 0}$ and $(\beta_t \mathbf{1}_{\{\psi(Z_t^s) > \epsilon_{00}/2\}})_{t\geq 0}$.

Finally, using the non-positivity of $(c_t)_{t\geq 0}$, we deduce

$$
\begin{aligned}
& d\left[\big(a + |m_t|^2 + t\varphi(\psi(Z_t^s))|\varsigma_t|^2\big)^{1/2} \exp\left(\int_0^t c_r dr\right)\right] \\
& \leq C\big(1 + t^{-1/2}\big)|\varsigma_t| \exp\left(\int_0^t c_r dr\right) dt + dn_t, \quad t \geq 0.
\end{aligned}
$$

Taking the expectation and letting a tend to 0, we complete the proof. □

Obviously, we wish to apply Lemma 8.5 with

$$
\varsigma_t = \hat{\zeta}_t^s, \ \Xi_t = \Xi(Z_t^s), \ c_t = \mathrm{Trace}\big[a_t D_{z,\bar{z}}^2 \psi(Z_t^s)\big],
$$

provided we have a bound for the term $\Xi(Z_t^s)$ in (8.28) and for ϵ_{00} to be fixed later on. (Basically, we cannot choose $\epsilon_{00} = 0$ since the coefficients driving the SDE satisfied by $(\hat{\zeta}_t^s)_{t\geq 0}$ are expected to be singular in the neighborhood of the boundary. See (8.1).)

As explained above, for the choice of Ξ we use below, the term $\Xi(Z_t^s)$ is bounded for Z_t^s away from the boundary of the domain only. Following Propositions 8.2 and 8.4, we are to localize the perturbation argument. Specifically,

Definition 8.6. *For some real $S > 0$, some smooth cut-off function $\phi : \mathbb{R}_+ \to [0,1]$ matching 1 on $[0,1]$ and 0 outside $[2,+\infty)$, some given positive real $\epsilon > 0$ and some (finite) stopping time $\mathfrak{s}$ at which $\psi(Z_{\mathfrak{s}}^s) > \epsilon$, we call* localized perturbation argument of Girsanov type *from time $\mathfrak{s}$ to time $\mathfrak{t} := \inf\{t > \mathfrak{s} : \psi(Z_t^s) \leq \epsilon\}$ ($\mathfrak{t}$ being possibly infinite) the perturbation of the Brownian motion $(B_t)_{t\geq 0}$ on the interval $[\mathfrak{s}, \mathfrak{t}]$ only. In such a case, the change of measure* *in* (8.26) *takes the form*

$$\mathbb{P}^\varepsilon(A) = \mathbb{E}\Big[\exp\Big(- \int_{\mathfrak{s}}^{t\wedge \mathfrak{t}} 2\mathrm{Re}\big[\langle \bar{G}(Z_r^s, \hat{Z}_r^{s+\varepsilon} - Z_r^s), dB_r\rangle\big] - \int_{\mathfrak{s}}^{t\wedge \mathfrak{t}} |G|^2(Z_r^s, \hat{Z}_r^{s+\varepsilon} - Z_r^s) dr\Big) \mathbf{1}_A\Big], \quad A \in \mathcal{F}_t,\ t \geq 0,$$

and the perturbed value function (with cut-off) in (8.27) *writes*

$$\begin{aligned}
&\hat{V}_S^{\sigma,\mathfrak{s},\mathfrak{t}}(s+\varepsilon) \\
&= \mathbb{E}\int_{\mathfrak{s}}^{\mathfrak{t}} \Big[\exp\Big(- \int_{\mathfrak{s}}^{t} 2\mathrm{Re}\big[\langle \bar{G}(Z_r^s, \hat{Z}_r^{s+\varepsilon} - Z_r^s), dB_r\rangle\big] \\
&\qquad - \int_{\mathfrak{s}}^{t} |G|^2(Z_r^s, \hat{Z}_r^{s+\varepsilon} - Z_r^s) dr\Big) \\
&\quad \times \exp\Big(\int_0^t \mathrm{Trace}[a_r D^2_{z,\bar{z}}\psi(\hat{Z}_r^{s+\varepsilon})] dr\Big) F\big(\det(a_t), a_t, \hat{Z}_t^{s+\varepsilon}\big)\phi\big(\frac{t}{T}\big)\Big] dt,
\end{aligned} \tag{8.30}$$

for some (progressively-measurable) extension of $(\hat{Z}_t^{s+\varepsilon})_{0\leq t\leq \mathfrak{s}}$ *to the time indices less than* $\mathfrak{s}$ *for which* $([d/d\varepsilon](\hat{Z}_t^{s+\varepsilon})_{|\varepsilon=0})_{0\leq t\leq \mathfrak{t}}$ *and* $([d^2/d\varepsilon^2](\hat{Z}_t^{s+\varepsilon})_{|\varepsilon=0})_{0\leq t\leq \mathfrak{t}}$ *exist. In such a case, by Lemma 8.5,*

$$\begin{aligned}
&\mathbb{E}\Big[\Big|\int_{\mathfrak{s}}^{t\wedge\mathfrak{t}} \langle \Xi(Z_r^s)\hat{\zeta}_r^s, dB_r\rangle\Big| \exp\Big(\int_0^{t\wedge\mathfrak{t}} \mathrm{Trace}[a_r D^2_{z,\bar{z}}\psi(Z_r^s)] dr\Big)\Big] \\
&\leq C\mathbb{E}\Big[\int_0^{t\wedge\mathfrak{t}} (1+r^{-1/2})|\hat{\zeta}_r^s| \exp\Big(\int_0^r \mathrm{Trace}[a_u D^2_{z,\bar{z}}\psi(Z_u^s)] du\Big) dr\Big],
\end{aligned}$$

for some constant $C > 0$*, only depending on* (**A**) *and on the bounds of* $(\Xi(Z_t^s))_{\mathfrak{s}\leq t\leq \mathfrak{t}}$ *and of the coefficients appearing in the Itô writing of* $(\zeta_t^s)_{0\leq t\leq \mathfrak{t}}$ *at times* $0 \leq t \leq \mathfrak{t}$ *for which* $\psi(Z_t^s) \geq \epsilon/2$*. (Pay attention that we here start from time* 0 *to benefit from* a *as initial condition in* (8.29)*.)*

We then deduce the analog of Proposition 8.2

Proposition 8.7. *Keep the assumptions of Definition 8.6 and assume that the function* Ξ *is bounded on the set* $\{\psi \geq \epsilon\}$ *If the differentiation operator w.r.t.* ε *and the expectation and integral symbols in the definition of* $\hat{V}_S^{\sigma,\mathfrak{s},\mathfrak{t}}$ *can be exchanged, then there exists a constant* $C > 0$*, only depending on Assumption* (**A**) *and on the bounds of* $(\Xi(\zeta_t^s))_{\mathfrak{s}\leq t\leq \mathfrak{t}}$ *and of the coefficients appearing in the Itô writing of* $(\zeta_t^s)_{0\leq t\leq \mathfrak{t}}$ *at times* $0 \leq t \leq \mathfrak{t}$ *for which* $\psi(Z_t^s) \geq \epsilon/2$*, such that*

$$\begin{aligned}
&\Big|\frac{d}{d\varepsilon}\big[\hat{V}^{\sigma,\mathfrak{s},\mathfrak{t}}(s+\varepsilon)\big]\Big| \\
&\leq C\mathbb{E}\Big[\int_{\mathfrak{s}}^{\mathfrak{t}} \exp\Big(\int_0^t \mathrm{Trace}[a_r D^2_{z,\bar{z}}(Z_t^s)] dr\Big)\Big[|\hat{\zeta}_t^s| + \int_0^t (1+r^{-1/2})|\hat{\zeta}_r^s| dr\Big] dt\Big],
\end{aligned}$$

where $\hat{\zeta}_t^s = [d/d\varepsilon](\hat{Z}_t^{s+\varepsilon})_{|\varepsilon=0}$.

Actually, the same strategy applies when differentiating twice in (8.30). It is then necessary to bound

$$\mathbb{E}\int_{\mathfrak{s}}^{\mathfrak{t}}\Big[\Big|\int_{\mathfrak{s}}^{t}\langle \Xi(Z_r^s)\hat{\zeta}_r^s, dB_r\rangle\Big|^2 \exp\Big(\int_0^t \mathrm{Trace}\big[a_r D_{z,\bar{z}}\psi(Z_r^s)\big]dr\Big)\Big]dt, \tag{8.31}$$

and

$$\mathbb{E}\int_{\mathfrak{s}}^{\mathfrak{t}}\Big|\int_{\mathfrak{s}}^{t}\langle \Xi(Z_r^s)\hat{\eta}_r^s, dB_r\rangle \exp\Big(\int_0^t \mathrm{Trace}\big[a_r D_{z,\bar{z}}\psi(Z_r^s)\big]dr\Big)\Big|dt, \tag{8.32}$$

with $\hat{\eta}_t^s = [d/d\varepsilon](\hat{Z}_t^{s+\varepsilon})_{|\varepsilon=0}$, and

$$\begin{aligned}&\mathbb{E}\int_{\mathfrak{s}}^{\mathfrak{t}}\Big|\int_{\mathfrak{s}}^{t}\langle\big(D_z\Xi(Z_r^s)\hat{\zeta}_r^s + D_{\bar{z}}\Xi(Z_r^s)\bar{\hat{\zeta}}_r^s\big)\hat{\zeta}_r^s, dB_r\rangle \\ &\quad\times \exp\Big(\int_0^t \mathrm{Trace}\big[a_r D_{z,\bar{z}}\psi(Z_r^s)\big]dr\Big)\Big|dt.\end{aligned} \tag{8.33}$$

For (8.32) and (8.33), the proof is the same as the one of Lemma 8.5. With the same notations as the ones used therein, the point is to consider (for $a > 0$)

$$d\Big[\big(a + |m_t|^2 + t\varphi(\psi(Z_t^s))[|\hat{\zeta}_t^s|^4 + |\hat{\eta}_t^s|^2]\big)^{1/2}\Big], \quad \mathfrak{s} \le t \le \mathfrak{t},$$

with

$$m_t = \int_{\mathfrak{s}}^{t}\langle \Xi(Z_r^s)\hat{\eta}_r^s, dB_r\rangle, \quad \mathfrak{s} \le t \le \mathfrak{t},$$

or

$$m_t = \int_0^t \langle\big(D_z\Xi(Z_r^s)\hat{\zeta}_r^s + D_{\bar{z}}\Xi(Z_r^s)\bar{\hat{\zeta}}_r\big)\hat{\zeta}_r^s, dB_r\rangle, \quad \mathfrak{s} \le t \le \mathfrak{t}.$$

For (8.31), it is sufficient to expand

$$\Big[\Big|\int_{\mathfrak{s}}^{t}\langle \Xi(Z_r^s)\hat{\zeta}_r^s, dB_r\rangle\Big|^2 \exp\Big(\int_0^t \mathrm{Trace}\big[a_r D_{z,\bar{z}}\psi(Z_r^s)\big]dr\Big)\Big]_{\mathfrak{s}\le t\le\mathfrak{t}}$$

by Itô's formula to get an analog of Lemma 8.5.

We then deduce

Proposition 8.8. *Keep the assumption Proposition 8.7. If the differentiation operator of order 2 w.r.t. ε and the expectation and integral symbols in the definition of $\hat{V}_S^{\sigma,\mathfrak{s},\mathfrak{t}}$ can be exchanged, then there exists a constant $C > 0$, only depending on Assumption (**A**) and on the bounds of $(\Xi(\zeta_t^s))_{\mathfrak{s}\le t\le\mathfrak{t}}$ and of the coefficients appearing in the Itô writing of $(\zeta_t^s)_{0\le t\le\mathfrak{t}}$ and $(\eta_t^s)_{0\le t\le\mathfrak{t}}$ at times $0 \le t \le \mathfrak{t}$ for which $\psi(Z_t^s) \ge \epsilon/2$, such that*

$$\begin{aligned}\Big|\frac{d^2}{d\varepsilon^2}\big[\hat{V}^{\sigma,\mathfrak{s},\mathfrak{t}}(s+\varepsilon)\big]\Big| \le C\mathbb{E}\Big[\int_{\mathfrak{s}}^{\mathfrak{t}}&\exp\Big(\int_0^t \mathrm{Trace}[a_r D^2_{z,\bar{z}}(Z_t^s)]dr\Big) \\ &\times\Big[|\hat{\eta}_t^s| + |\hat{\zeta}_t^s|^2 + \int_0^t (1 + r^{-1/2})|\hat{\eta}_r^s|dr \\ &\quad + \int_0^t (1 + r^{-1/2})|\hat{\zeta}_r^s|^2 dr + \Big(\int_0^t |\hat{\zeta}_r^s| dr\Big)^2\Big]dt\Big],\end{aligned}$$

where $\hat{\eta}_t^s = [d^2/d\varepsilon^2](\hat{Z}_t^{s+\varepsilon})_{|\varepsilon=0}$.

8.5. **Explicit Computations at the Boundary.** We are now in position to expand the computations. We start with the so-called "enlargement of the set of controls" method. Following the localization procedure described in the statement of Proposition 8.2, the time indices t we consider below are always assumed to belong to the interval $[\mathfrak{s},\mathfrak{t}]$, the choice of the parameter ϵ in Proposition 8.2 being clearly specified at the end of the discussion. Recall that for $t\in[\mathfrak{s},\mathfrak{t}]$, $\psi(Z_t^s)$ is less than ϵ. Recall also from (8.10) that the perturbation reads

$$
\begin{aligned}
d\hat{Z}_t^{s+\varepsilon} &= \psi^{1/2}(\hat{Z}_t^{s+\varepsilon})\exp\bigl(P(Z_t^s,\hat{Z}_t^{s+\varepsilon}-Z_t^s)\bigr)\sigma_t dB_t \\
&\quad + \exp\bigl(P(Z_t^s,\hat{Z}_t^{s+\varepsilon}-Z_t^s)\bigr)a_t\exp\bigl(\bar{P}^*(Z_t^s,\hat{Z}_t^{s+\varepsilon}-Z_t^s)\bigr)D_{\bar{z}}^*\psi(\hat{Z}_t^{s+\varepsilon})dt,
\end{aligned}
\tag{8.34}
$$

where $t\in[\mathfrak{s},\mathfrak{t}]$, and (see (8.11))

$$
\begin{aligned}
&\frac{d}{d\varepsilon}\bigl[P(Z_t^s,\hat{Z}_t^{s+\varepsilon}-Z_t^s)\bigr] \\
&= |D_z\psi_t|^{-2}\bigl[D_{\bar{z},z}^2\psi_t\zeta_t D_z\psi_t + D_{\bar{z},\bar{z}}^2\psi_t\bar{\zeta}_t D_z\psi_t \\
&\qquad - D_{\bar{z}}^*\psi_t(D_{z,\bar{z}}^2\psi_t\bar{\zeta}_t)^* - D_{\bar{z}}^*\psi_t(D_{z,z}^2\psi_t\zeta_t)^*\bigr], \\
&:= Q_t\zeta_t, \quad t\in[\mathfrak{s},\mathfrak{t}],
\end{aligned}
\tag{8.35}
$$

ζ_t being given by $\zeta_t=[d/d\varepsilon][Z_t^{s+\varepsilon}]$, $t\in[\mathfrak{s},\mathfrak{t}]$.

We emphasize that (8.35) makes sense for ϵ small enough: since $\psi(Z_t^s)\leq\epsilon$ for $t\in[\mathfrak{s},\mathfrak{t}]$, $|D_z\psi_t(Z_t^s)|\neq 0$ for ϵ small enough and $t\in[\mathfrak{s},\mathfrak{t}]$.

We also make use of the following abbreviated notation: we get rid of the symbol hat "^" and of the superscript s for more simplicity in $(\hat{\zeta}_t^s)_{\mathfrak{s}\leq t\leq\mathfrak{t}}$ (compare with the statement of Proposition 8.2); we also write ψ_t for $\psi(Z_t^s)$ and $L\psi_t$ for $\mathrm{Trace}[a_t D_{z,\bar{z}}^2\psi(Z_t^s)]$, $\mathfrak{s}\leq t\leq\mathfrak{t}$.

We then write the derivative $(\zeta_t)_{\mathfrak{s}\leq t\leq\mathfrak{t}}$ as the solution of[20]

$$
\begin{aligned}
d\zeta_t &= \bigl\{\psi_t^{-1/2}\mathrm{Re}\bigl[D_z\psi_t\zeta_t\bigr]+\psi_t^{1/2}Q_t\zeta_t\bigr\}\sigma_t dB_t \\
&\quad + \bigl[a_tD_{\bar{z},z}\psi_t\zeta_t + a_tD_{\bar{z},\bar{z}}\psi_t\bar{\zeta}_t\bigr]dt + \bigl[Q_t\zeta_t a_t D_{\bar{z}}^*\psi_t - a_tQ_t\zeta_tD_{\bar{z}}^*\psi_t\bigr]dt.
\end{aligned}
$$

Above, the vector $(\sum_{j,k=1}^d (a_t)_{i,j}D^2_{\bar{z}_j,z_k}\psi(Z_t^s)(\zeta_t)_k)_{1\leq i\leq d}$ is represented by the product $a_tD_{\bar{z},z}\psi_t\zeta_t$.

From (8.35), we have (pay attention that $D_z\psi_t a_t D_{\bar{z}}^*\psi_t$ and $[(D_{z,\bar{z}}^2\psi_t\bar{\zeta}_t)^*+(D_{z,z}^2\psi_t\zeta_t)^*]D_{\bar{z}}^*\psi_t$ below stand for scalar quantities as products of row and colum vectors)

$$
\begin{aligned}
&a_tD_{\bar{z},z}\psi_t\zeta_t + a_tD_{\bar{z},\bar{z}}\psi_t\bar{\zeta}_t + Q_t\zeta_t a_tD_{\bar{z}}^*\psi_t - a_tQ_t\zeta_tD_{\bar{z}}^*\psi_t \\
&= |D_z\psi_t|^{-2}D_z\psi_t a_t D_{\bar{z}}^*\psi_t\bigl(D_{\bar{z},z}^2\psi_t\zeta_t + D_{\bar{z},\bar{z}}^2\psi_t\bar{\zeta}_t\bigr) \\
&\quad - |D_z\psi_t|^{-2}D_{\bar{z}}^*\psi_t\bigl[(D_{z,\bar{z}}^2\psi_t\bar{\zeta}_t)^*+(D_{z,z}^2\psi_t\zeta_t)^*\bigr]a_tD_{\bar{z}}^*\psi_t \\
&\quad + |D_z\psi_t|^{-2}\bigl[(D_{z,\bar{z}}^2\psi_t\bar{\zeta}_t)^*+(D_{z,z}^2\psi_t\zeta_t)^*\bigr]D_{\bar{z}}^*\psi_t a_t D_{\bar{z}}^*\psi_t \\
&:= |D_z\psi_t|^{-2}D_z\psi_t a_t D_{\bar{z}}^*\psi_t\bigl(D_{\bar{z},z}^2\psi_t\zeta_t + D_{\bar{z},\bar{z}}^2\psi_t\bar{\zeta}_t\bigr) + H_t a_t D_{\bar{z}}^*\psi_t,
\end{aligned}
\tag{8.36}
$$

$(H_t)_{\mathfrak{s}\leq t\leq\mathfrak{t}}$ here standing for the auxiliary process

$$
\begin{aligned}
H_t &= |D_z\psi_t|^{-2}\bigl\{-D_{\bar{z}}^*\psi_t\bigl[(D_{z,\bar{z}}^2\psi_t\bar{\zeta}_t)^*+(D_{z,z}^2\psi_t\zeta_t)^*\bigr] \\
&\qquad + \bigl[(D_{z,\bar{z}}^2\psi_t\bar{\zeta}_t)^*+(D_{z,z}^2\psi_t\zeta_t)^*\bigr]D_{\bar{z}}^*\psi_t\bigr\},
\end{aligned}
\tag{8.37}
$$

with values in $\mathbb{C}^{d\times d}$.

[20]Again, the differentiation is purely formal since no differentiability property has been established yet. This is the so-called "meta" part of Meta-Theorem 8.1.

We deduce that

$$\begin{aligned}
d\zeta_t = &\big\{\psi_t^{-1/2}\mathrm{Re}\big[D_z\psi_t\zeta_t\big] + \psi_t^{1/2}Q_t\zeta_t\big\}\sigma_t dB_t \\
&+ |D_z\psi_t|^{-2}D_z\psi_t a_t D_{\bar z}^*\psi_t\big(D^2_{\bar z,z}\psi_t\zeta_t + D^2_{\bar z,\bar z}\psi_t\bar\zeta_t\big)dt + H_t a_t D^*_{\bar z}\psi_t dt.
\end{aligned}$$

Taking the square norm, we obtain

$$\begin{aligned}
&d|\zeta_t|^2 \\
&= 2|D_z\psi_t|^{-2}D_z\psi_t a_t D^*_{\bar z}\psi_t \mathrm{Re}\big[\langle\bar\zeta_t, \big(D^2_{\bar z,z}\psi_t\zeta_t + D^2_{\bar z,\bar z}\psi_t\bar\zeta_t\big)\big]dt \\
&\quad + 2\mathrm{Re}\big[\langle\bar\zeta_t, H_t a_t D^*_{\bar z}\psi_t\rangle\big]dt \\
&\quad + \mathrm{Trace}\big[\big(\psi_t^{-1/2}\mathrm{Re}\big[D_z\psi_t\zeta_t\big]I_d + \psi_t^{1/2}Q_t\zeta_t\big) \\
&\qquad\quad \times a_t\big(\psi_t^{-1/2}\mathrm{Re}\big[D_z\psi_t\zeta_t\big]I_d - \psi_t^{1/2}\big(Q_t\zeta_t\big)^*\big)\big]dt \\
&\quad + \psi_t^{-1/2}\mathrm{Re}\big[D_z\psi_t\zeta_t\big]\langle\bar\zeta_t, \sigma_t dB_t\rangle + \psi_t^{-1/2}\mathrm{Re}\big[D_z\psi_t\zeta_t\big]\langle\zeta_t, \bar\sigma_t d\bar B_t\rangle \\
&\quad + \psi_t^{1/2}\big[\langle\bar\zeta_t, Q_t\zeta_t\sigma_t dB_t\rangle + \langle\zeta_t, \overline{Q_t\zeta_t}\bar\sigma_t d\bar B_t\rangle\big], \quad \mathfrak{s} \le t \le \mathfrak{t}.
\end{aligned} \tag{8.38}$$

In what follows, we modify the choice of ψ according to the following observation: for any constant $c > 0$, $c\psi$ is again a plurisuperharmonic function describing the domain. To make things clear, we denote by ψ^0 some *reference plurisuperharmonic function* such that, for any Hermitian matrix a of trace 1 and for any $z \in \mathcal{D}$, $\mathrm{Trace}[aD^2_{z,\bar z}\psi^0(z)] \le -1$. Then, we understand ψ as $N\psi^0$ for some free parameter $N \ge 1$ that will be fixed later on.

As a first application, we can simplify the form of $d|\zeta_t|^2$, or at least we can bound it. As already said, for $\epsilon > 0$ small, $\psi^0_t \le N\psi^0_t \le \epsilon$, $t \in [\mathfrak{s}, \mathfrak{t}]$, so that $|D_z\psi^0_t| \ge \kappa$ for some given constant $\kappa > 0$, $\mathfrak{s} \le t \le \mathfrak{t}$. For example, we notice that $|Q_t\zeta_t|$ in (8.35) and $|H_t|$ in (8.37) by can be bounded by $C|\zeta_t|$, i.e.

$$|Q_t\zeta_t|, |H_t| \le C|\zeta_t|, \quad \mathfrak{s} \le t \le \mathfrak{t}, \tag{8.39}$$

for some constant C depending on κ, $\|D\psi^0\|_\infty$ and $\|D^2\psi^0\|_\infty$, but independent of N. Therefore, denoting by $(r_t)_{\mathfrak{s}\le t\le\mathfrak{t}}$ a generic bounded process, bounded by some constant C at any time in $[\mathfrak{s}, \mathfrak{t}]$, and setting $\mathcal{E}^0_t := D_z\psi^0_t a_t D^*_{\bar z}\psi^0_t$, we write

$$\begin{aligned}
&d|\zeta_t|^2 \\
&= \psi_t^{-1}\mathrm{Re}^2\big[D_z\psi_t\zeta_t\big]dt + \mathrm{Re}\big[D_z\psi_t\zeta_t\big]|\zeta_t|r_t dt + \psi_t|\zeta_t|^2 r_t dt \\
&\quad + N|\zeta_t|^2\big((\mathcal{E}^0_t)^{1/2} + \mathcal{E}^0_t\big)r_t dt \\
&\quad + \psi_t^{-1/2}\mathrm{Re}\big[D_z\psi_t\zeta_t\big]\langle\bar\zeta_t, \sigma_t dB_t\rangle + \psi_t^{-1/2}\mathrm{Re}\big[D_z\psi_t\zeta_t\big]\langle\zeta_t, \bar\sigma_t d\bar B_t\rangle \\
&\quad + \psi_t^{1/2}\big[\langle\bar\zeta_t, Q_t\zeta_t\sigma_t dB_t\rangle + \langle\zeta_t, \overline{Q_t\zeta_t}\bar\sigma_t d\bar B_t\rangle\big], \quad \mathfrak{s} \le t \le \mathfrak{t},
\end{aligned} \tag{8.40}$$

the constant C in the bound of $(r_t)_{\mathfrak{s}\le t\le\mathfrak{t}}$ depending on $(\mathbf{A})$ only (and not on N). In particular, C may depend on κ. (Above, the writing $((\mathcal{E}^0_t)^{1/2} + \mathcal{E}^0_t)r_t$ is an abuse of notation. It stands for $(\mathcal{E}^0_t)^{1/2}r_t + \mathcal{E}^0_t r_t$ for possibly different values of r. We will use this simplification quite often below.) One way or another, we understand that the terms $(\psi_t^{-1}\mathrm{Re}^2[D_z\psi_t\zeta_t])_{t\ge0}$ and $(\mathcal{E}^0_t)_{t\ge0}$ are to be controlled to control the *derivative quantity* according to the program announced in Section 7.

The strategy we here develop (and inspired by the one of Krylov) consists in considering a modified version of the *derivative quantity*. Below, we consider

$$\bar{\Gamma}_t = \exp(-K\psi_t)|\zeta_t|^2 + \psi_t^{-1}\mathrm{Re}^2\big[D_z\psi_t\zeta_t\big], \quad \mathfrak{s} \leq t \leq \mathfrak{t}, \tag{8.41}$$

for some constant $K>0$ to be chosen later on.

To compute $(d\bar{\Gamma}_t)_{\mathfrak{s}\leq t\leq \mathfrak{t}}$, we use the following writing for $(d\psi_t)_{\mathfrak{s}\leq t\leq\mathfrak{t}}$

$$d\psi_t = \psi_t^{1/2}\big[D_z\psi_t\sigma_t dB_t + D_{\bar{z}}\psi_t\bar{\sigma}_t d\bar{B}_t\big] + 2D_z\psi_t a_t D_{\bar{z}}^*\psi_t dt + \psi_t L\psi_t dt, \quad \mathfrak{s}\leq t\leq \mathfrak{t}. \tag{8.42}$$

(Apply Itô's formula to $(\psi(Z_t^s))_{\mathfrak{s}\leq t\leq\mathfrak{t}}$ and have in mind that $P(Z^s,\hat{Z}^s-Z^s)=0$ when $\hat{Z}^s$ in (8.34) is Z^s itself.) We first write

$$\begin{aligned}
&d\exp(-K\psi_t)\\
&= -2K\exp(-K\psi_t)\psi_t^{1/2}\mathrm{Re}\big[D_z\psi(Z_t^s)\sigma dB_t\big]\\
&\quad + [K^2\psi_t - 2K]\exp(-K\psi_t)\langle D_z\psi_t, a_t D_{\bar{z}}\psi_t\rangle dt\\
&\quad - K\exp(-K\psi_t)\psi_t L\psi_t dt\\
&= -2K\exp(-K\psi_t)\psi_t^{1/2}\mathrm{Re}\big[D_z\psi(Z_t^s)\sigma dB_t\big]\\
&\quad + N^2[K^2\psi_t - 2K]\exp(-K\psi_t)\mathcal{E}_t^0 dt - NK\exp(-K\psi_t)\psi_t L\psi_t^0 dt.
\end{aligned} \tag{8.43}$$

Using (8.40),

$$\begin{aligned}
&d\big[\exp(-K\psi_t)|\zeta_t|^2\big]\\
&= \exp(-K\psi_t)\big[\psi_t^{-1}\mathrm{Re}^2[D_z\psi_t\zeta_t] + \mathrm{Re}[D_z\psi_t\zeta_t]|\zeta_t|r_t\\
&\quad + \psi_t|\zeta_t|^2 r_t + N|\zeta_t|^2\big((\mathcal{E}_t^0)^{1/2} + \mathcal{E}_t^0\big)r_t\big]dt\\
&\quad + |\zeta_t|^2\exp(-K\psi_t)\big[N^2[K^2\psi_t - 2K]\mathcal{E}_t^0 - NK\psi_t L\psi_t^0\big]dt\\
&\quad + NK\exp(-K\psi_t)\big[\mathrm{Re}[D_z\psi_t\zeta_t]|\zeta_t| + \psi_t|\zeta_t|^2\big]r_t + dm_t, \quad \mathfrak{s}\leq t\leq\mathfrak{t},
\end{aligned}$$

where $(m_t)_{t\geq 0}$ stands for a generic martingale term. We are now in position to compute $d\bar{\Gamma}_t$ at any time $t\in[\mathfrak{s},\mathfrak{t}]$. Have in mind that, for such t's, ψ_t is less than ϵ and $(r_t)_{\mathfrak{s}\leq t\leq\mathfrak{t}}$ is a generic process satisfying $|r_t|\leq C$, for some C depending on $(\mathbf{A})$ only. Think in particular of the useful bound: $|\mathrm{Re}[D_z\psi_t\zeta_t]| \leq \epsilon^{1/2}\psi_t^{-1/2}|\mathrm{Re}[D_z\psi_t\zeta_t]|$, $t\in[\mathfrak{s},\mathfrak{t}]$. Then, applying Young's inequality to the term $N(\mathcal{E}_t^0)^{1/2}$, the above equation has the form

$$\begin{aligned}
&d\big[\exp(-K\psi_t)|\zeta_t|^2\big]\\
&\leq \exp(-K\psi_t)\big[\psi_t^{-1}\mathrm{Re}^2[D_z\psi_t\zeta_t] + C(1+\epsilon^{1/2}+\epsilon)|\xi_t|^2\\
&\qquad\qquad + C(N+N^2)|\zeta_t|^2\mathcal{E}_t^0\big]dt\\
&\quad + |\zeta_t|^2\exp(-K\psi_t)\big[N^2[K^2\epsilon - 2K]\mathcal{E}_t^0 + CNK\epsilon\big]dt\\
&\quad + NK\exp(-K\psi_t)\big[C\epsilon^{1/2}|\xi_t|^2 + C\epsilon|\xi_t|^2\big] + dm_t,
\end{aligned} \tag{8.44}$$

where $|\xi_t|^2 = |\zeta_t|^2 + \psi_t^{-1}\mathrm{Re}^2[D_z\psi_t\zeta_t]$. To complete the analysis (in the neighborhood of the boundary), we must compute $d[\psi_t^{-1}\mathrm{Re}^2[D_z\psi_t\zeta_t]]$, $\mathfrak{s}\leq t\leq\mathfrak{t}$. To do so, we start with (8.42) at $s+\varepsilon$ (so that a_t is understood as $\exp(p_t^\varepsilon)a_t\exp(-p_t^\varepsilon)$). Taking the square root, we write

$$
\begin{aligned}
& d\psi^{1/2}(Z_t^{s+\varepsilon}) \\
& = \frac{1}{2}\big[D_z\psi(Z_t^{s+\varepsilon})\exp(p_t^\varepsilon)\sigma_t dB_t + D_{\bar z}\psi(Z_t^{s+\varepsilon})\exp(\bar p_t^\varepsilon)\bar\sigma_t d\bar B_t\big] \\
& \quad + \frac{3}{4}\psi^{-1/2}(Z_t^{s+\varepsilon})D_z\psi(Z_t^{s+\varepsilon})\exp(p_t^\varepsilon)a_t\exp(-p_t^\varepsilon)D_{\bar z}^*\psi(Z_t^{s+\varepsilon})dt \\
& \quad + \frac{1}{2}\psi^{1/2}(Z_t^{s+\varepsilon})\mathrm{Trace}\big[\exp(p_t^\varepsilon)a_t\exp(-p_t^\varepsilon)D_{z,\bar z}^2\psi(Z_t^{s+\varepsilon})\big]dt.
\end{aligned}
$$

We now differentiate with respect to ε at $\varepsilon=0$. We obtain (with the notation $\mathcal{E}_t = D_z\psi_t a_t D_{\bar z}^*\psi_t = N^2\mathcal{E}_t^0$)

$$
\begin{aligned}
& \frac{1}{2}d\big[\psi_t^{-1/2}\mathrm{Re}[D_z\psi(Z_t)\zeta_t]\big] \\
& = \mathrm{Re}\big[\big((D_{z,z}\psi_t\zeta_t)^* + (D_{z,\bar z}\psi_t\bar\zeta_t)^* + D_z\psi_t Q_t\zeta_t\big)\sigma_t dB_t\big] \\
& \quad - \frac{3}{4}\psi_t^{-3/2}\mathrm{Re}[D_z\psi_t\zeta_t]\mathcal{E}_t dt \\
& \quad + \frac{3}{4}\psi_t^{-1/2}\big[\big((D_{z,z}\psi_t\zeta_t)^* + (D_{z,\bar z}\psi_t\bar\zeta_t)^* + D_z\psi_t Q_t\zeta_t\big)a_t D_{\bar z^*}\psi_t\big]dt \\
& \quad + \frac{3}{4}\psi_t^{-1/2}\big[D_z\psi_t a_t\big(D_{\bar z,z}\psi_t\zeta_t + D_{\bar z,\bar z}\psi_t\bar\zeta_t - Q_t\zeta_t D_{\bar z}^*\psi_t\big)\big]dt \\
& \quad + \frac{1}{2}\psi_t^{1/2}\mathrm{Trace}\big[\big(Q_t\zeta_t a_t - a_t Q_t\zeta_t\big)D_{z,\bar z}^2\psi_t + a_t D_{z,\bar z,z}^2\psi_t\zeta_t + a_t D_{z,\bar z,\bar z}^2\psi_t\bar\zeta_t\big]dt \\
& \quad + \frac{1}{2}\mathrm{Re}[D_z\psi_t\zeta_t]\psi_t^{-1/2}L\psi_t dt.
\end{aligned}
\tag{8.45}
$$

Plugging the definition of $(Q_t\zeta_t)_{\mathfrak{s}\le t\le\mathfrak{t}}$ (see (8.35)), we deduce

$$
\begin{aligned}
& (D_{z,z}\psi_t\zeta_t)^* + (D_{z,\bar z}\psi_t\bar\zeta_t)^* + D_z\psi_t Q_t\zeta_t \\
& = |D_z\psi_t|^{-2}\big(D_z\psi_t D_{\bar z,z}^2\psi_t\zeta_t + D_z\psi_t D_{\bar z,\bar z}^2\psi_t\bar\zeta_t\big)D_z\psi_t \\
& = r_t|\zeta_t|D_z\psi_t.
\end{aligned}
\tag{8.46}
$$

It is important to notice that the process $(r_t)_{\mathfrak{s}\le t\le\mathfrak{t}}$ in (8.46) is scalar as the product of row and column vectors. (It is also bounded independently of N.) We deduce

$$
\begin{aligned}
& d\big[\psi_t^{-1/2}\mathrm{Re}[D_z\psi_t\zeta_t]\big] \\
& = 2\mathrm{Re}\big[r_t|\zeta_t|D_z\psi_t\sigma_t dB_t\big] - \frac{3}{2}\psi_t^{-3/2}\mathrm{Re}[D_z\psi_t\zeta_t]\mathcal{E}_t dt \\
& \quad + \psi_t^{-1/2}r_t\mathcal{E}_t|\zeta_t|dt + N\psi_t^{1/2}r_t|\zeta_t|dt + \mathrm{Re}[D_z\psi_t\zeta_t]\psi_t^{-1/2}L\psi_t dt.
\end{aligned}
$$

Taking the square, we finally claim (use the following trick to pass from the equality to the inequality : $\psi_t^{-1}r_t|\zeta_t|\mathrm{Re}[D_z\psi_t\zeta_t]\mathcal{E}_t \le \psi_t^{-2}\mathrm{Re}^2[D_z\psi_t\zeta_t]\mathcal{E}_t + r_t^2|\zeta_t|^2\mathcal{E}_t$, $Nr_t|\zeta_t|\mathrm{Re}[D_z\psi_t\zeta_t] \le$

$N\psi_t r_t^2|\zeta_t|^2 + N\psi_t^{-1}\mathrm{Re}^2[D_z\psi_t\zeta_t]$ and $L\psi_t \le -N$)

$$
\begin{aligned}
&d\big[\psi_t^{-1}\mathrm{Re}^2[D_z\psi_t\zeta_t]\big]\\
&= dm_t + r_t|\zeta_t|^2\mathcal{E}_t dt - 3\psi_t^{-2}\mathrm{Re}^2[D_z\psi_t\zeta_t]\mathcal{E}_t dt + \psi_t^{-1}r_t|\zeta_t|\mathrm{Re}[D_z\psi_t\zeta_t]\mathcal{E}_t dt\\
&\quad + Nr_t|\zeta_t|\mathrm{Re}[D_z\psi_t\zeta_t]dt + 2\psi_t^{-1}\mathrm{Re}^2[D_z\psi_t\zeta_t]L\psi_t dt\\
&\le dm_t + C(1+\mathcal{E}_t)|\zeta_t|^2 dt + CN\psi_t|\zeta_t|^2 dt - N\psi_t^{-1}\mathrm{Re}^2[D_z\psi_t\zeta_t]dt,
\end{aligned} \tag{8.47}
$$

for a possibly new value of C.

Making the sum with (8.44) and assuming $\epsilon < 1$ and $N \ge 1$, we deduce

$$
\begin{aligned}
d\bar{\Gamma}_t &\le \exp(-K\psi_t)(1-N)\psi_t^{-1}\mathrm{Re}^2[D_z\psi_t\zeta_t]dt\\
&\quad + |\xi_t|^2\big(C' + C'N\epsilon^{1/2} + C'NK\epsilon^{1/2}\big)dt\\
&\quad + |\zeta_t|^2\exp(-K\psi_t)N^2\big[K^2\epsilon - 2K + C'\exp(K\psi_t)\big]\mathcal{E}_t^0 dt + dm_t,
\end{aligned}
$$

the constant C' depending on C only. (In particular, C' is independent of K, N, ϵ, s and t.)

Choose now $K = \epsilon^{-1/4}$. We obtain

$$
\begin{aligned}
d\bar{\Gamma}_t &\le \exp(-K\psi_t)(1-N)\psi_t^{-1}\mathrm{Re}^2[D_z\psi_t\zeta_t]dt + 2(C' + 2C'N\epsilon^{1/4})|\xi_t|^2 dt\\
&\quad + |\zeta_t|^2\exp(-K\psi_t)N^2\big[\epsilon^{1/2} - 2\epsilon^{-1/4} + C'\exp(\epsilon^{3/4})\big]\mathcal{E}_t^0 dt + dm_t.
\end{aligned}
$$

Choose ϵ small enough such that

$$
\epsilon^{1/2} - 2\epsilon^{-1/4} + C'\exp(\epsilon^{3/4}) < 0. \tag{8.48}
$$

Then,

$$
d\bar{\Gamma}_t \le \exp(-K\psi_t)(1-N)\psi_t^{-1}\mathrm{Re}^2[D_z\psi_t\zeta_t]dt + 2\big(C' + 2C'N\epsilon^{1/4}\big)|\xi_t|^2 dt + dm_t,
$$

for $\mathfrak{s} \le t \le \mathfrak{t}$. Finally for $N = \epsilon^{-1/4}$ and $\exp(\epsilon^{3/4}) \le 2$, we obtain:

$$
d\bar{\Gamma}_t \le 6C'|\xi_t|^2 dt + dm_t \le 6C'\exp(\epsilon^{3/4})\bar{\Gamma}_t + dm_t \le 12C'\bar{\Gamma}_t + dm_t, \tag{8.49}
$$

for $\mathfrak{s} \le t \le \mathfrak{t}$.

Exactly as in the statement of Proposition 7.7 (see in particular (7.17)), the right quantity to consider is

$$
\exp\Big(\int_0^t L\psi_r dr\Big)\bar{\Gamma}_t = \exp\Big(\int_0^t NL\psi_r^0 dr\Big)\bar{\Gamma}_t, \quad \mathfrak{s} \le t \le \mathfrak{t}.
$$

Again, for $\mathfrak{s} \le t \le \mathfrak{t}$,

$$
\begin{aligned}
&d\Big[\exp\Big(\int_0^t NL\psi_r^0 dr\Big)\bar{\Gamma}_t\Big]\\
&\le \exp\Big(\int_0^t NL\psi_r^0 dr\Big)\Big[NL\psi_t^0\bar{\Gamma}_t + 12C'\bar{\Gamma}_t\Big]dt + dm_t\\
&\le \exp\Big(\int_0^t NL\psi_r^0 dr\Big)\Big[-N\bar{\Gamma}_t + 12C'\bar{\Gamma}_t\Big]dt + dm_t.
\end{aligned}
$$

Having in mind that $N = \epsilon^{-1/4}$, we deduce that, for $\epsilon^{-1/4} \ge 12C'$ (obviously, this is compatible with (8.48)),

$$
d\Big[\exp\Big(\int_0^t NL\psi_r^0 dr\Big)\bar{\Gamma}_t\Big] \le dm_t, \quad \mathfrak{s} \le t \le \mathfrak{t}. \tag{8.50}
$$

Actually, it is plain to see that, for ϵ small enough, the same holds with $NL\psi_s^0$ replaced by $(N-1)L\psi_s^0$, i.e.

$$d\left[\exp\left(\int_0^t (N-1)L\psi_s^0 ds\right)\bar{\Gamma}_t\right] \le dm_t, \quad \mathfrak{s} \le t \le \mathfrak{t}. \tag{8.51}$$

We deduce

Proposition 8.9. *There exists a positive real ϵ_1 such that for $0 < \epsilon < \epsilon_1$, for $N = K = \epsilon^{-1/4}$, for $\psi = N\psi^0$, where ψ^0 is the* reference plurisuperharmonic function *describing $\mathcal{D}$ such that* $\mathrm{Trace}[aD^2_{z,\bar{z}}\psi^0(z)] \le -1$, *$z \in \mathcal{D}$, for a stopping time $\mathfrak{s}$ at which $\psi(Z^s_{\mathfrak{s}}) < \epsilon$, the* derivative quantity *obtained by perturbing the control parameter as in* (8.10) *and* (8.35)

$$\bar{\Gamma}_t^{(1)} = \exp\bigl(-K\psi(Z_t^s)\bigr)|\zeta_t|^2 + \psi^{-1}(Z_t^s)\mathrm{Re}^2\bigl[D_z\psi(Z_t^s)\zeta_t\bigr], \quad t \ge \mathfrak{s},$$

satisfies up to time $\mathfrak{t} = \inf\{t \ge \mathfrak{s} : \psi(Z_t^s) \ge \epsilon\}$ (provided that $(\hat{Z}_t^{s+\varepsilon})_{0\le t \le \mathfrak{t}}$ is well differentiable w.r.t. ε)

$$\begin{aligned}&\mathbb{E}\left[\exp\left(\int_0^{t\wedge\mathfrak{t}} (1-\delta)\mathrm{Trace}[a_r D^2_{z,\bar{z}}\psi(Z_r^s)]dr\right)\bar{\Gamma}^{(1)}_{t\wedge\mathfrak{t}}|\mathcal{F}_{\mathfrak{s}}\right] \\ &\le \exp\left(\int_0^{\mathfrak{s}} (1-\delta)\mathrm{Trace}[a_r D^2_{z,\bar{z}}\psi(Z_r^s)]dr\right)\bar{\Gamma}^{(1)}_{\mathfrak{s}}, \quad t \ge \mathfrak{s},\end{aligned}$$

with $\delta = 1/N = \epsilon^{1/4}$.

8.6. **Away from the Boundary.** We now investigate the *derivative quantity* away from the boundary. The idea consists in perturbing the system in two different ways as the same time, or said differently, in applying two perturbations. In the subsections above, this possiblity has not been discussed, but we feel it quite simple to understand: it is even plain to see that provided that the corresponding versions of Propositions 8.2, 8.4 or 8.7 be true for each perturbation under consideration, the common action of both perturbations on the perturbed value function is of the same type, i.e. the statements of Propositions 8.2, 8.4 or 8.7 (according to the framework) remain true under the common action.

Away from the boundary, the idea is to perturb both the underlying time speed, as explained in Subsection 8.3, and the probability measure, as explained in Subsection 8.4.

Following the localization procedure described in the statement of Propositions 8.4 and 8.7, the time indices t we consider in this subsection are always assumed to belong to the interval $[\mathfrak{s}, \mathfrak{t}]$, where $\mathfrak{s}$ is some stopping time at which $\psi(Z_t^s) > \epsilon$[21] and $\mathfrak{t} = \inf\{t > \mathfrak{s} : \psi(Z_t^s) \le \epsilon\}$. (As above, the choice of the parameter ϵ is clearly specified at the end of the discussion.) In particular for $t \in [\mathfrak{s}, \mathfrak{t}]$, $\psi(Z_t^s)$ is greater than ϵ.

We also make use of the same abbreviated notation as above: we get rid of the symbol hat "$\hat{}$" and of the superscript s for more simplicity in $(\hat{\zeta}_t^s)_{\mathfrak{s}\le t\le\mathfrak{t}}$; we also write ψ_t for $\psi(Z_t^s)$ and $L\psi_t$ for $\mathrm{Trace}[a_t D^2_{z,\bar{z}}\psi(Z_t^s)]$, $\mathfrak{s} \le t \le \mathfrak{t}$. Finally, we emphasize that ψ is here arbitrary: the connection with the form $\psi = N\psi^0$ used in Proposition 8.9 is explained later on.

The time-change we here use is given by a variation of (8.20), namely

$$\frac{d}{d\varepsilon}\bigl[T(Z_t^s, \hat{Z}_t^{s+\varepsilon} - Z_t^s)\bigr]_{|\varepsilon=0} = -\psi^{-1}(Z_t^s)\mathrm{Re}\bigl[D_z\psi(Z_t^s)\zeta_t\bigr], \quad \mathfrak{s} \le t \le \mathfrak{t}. \tag{8.52}$$

[21]Pay attention that the values of ϵ may be different from the ones given by Proposition 8.9.

Moreover, the measure perturbation we choose in (8.28) is

$$\frac{d}{d\varepsilon}\big[G(Z_t^s, \hat{Z}_t^{s+\varepsilon} - Z_t^s)\big]_{|\varepsilon=0} = -\Lambda \bar{\sigma}_t^* \zeta_t, \quad \mathfrak{s} \leq t \leq \mathfrak{t}, \tag{8.53}$$

for some constant Λ to be chosen. (In other words, $\Xi(Z_t^s) = -\Lambda \bar{\sigma}_t^*$ in (8.28).)

We emphasize that both perturbations (8.52) and (8.53) are linear functionals of ζ, with a bounded linear coefficient. (Again, $\psi^{-1}(Z_t^s)$ is bounded by ϵ^{-1} away for $t \in [\mathfrak{s}, \mathfrak{t}]$.)

The dynamics of $(\hat{Z}_t^{s+\varepsilon})_{\mathfrak{s}\leq t\leq \mathfrak{t}}$ then read (compare with (8.18) and (8.25))

$$\begin{aligned} d\hat{Z}_t^{s+\varepsilon} &= \psi^{1/2}(\hat{Z}_t^{s+\varepsilon}) T(Z_t^s, \hat{Z}_t^{s+\varepsilon} - Z_t^s)\sigma_t \big[dB_t + G(Z_t^s, \hat{Z}_t^{s+\varepsilon} - Z_t^s)dt\big] \\ &\quad + T^2(Z_t^s, \hat{Z}_t^{s+\varepsilon} - Z_t^s) a_t D_{\bar{z}}^* \psi(Z_t^{s+\varepsilon})dt, \quad \mathfrak{s} \leq t \leq \mathfrak{t}. \end{aligned}$$

Differentiating (at least formally), we obtain

$$d\zeta_t = -\Lambda a_t \zeta_t dt + a_t D_{\bar{z},z}^* \psi_t \zeta_t dt + a_t D_{\bar{z},\bar{z}}^* \psi_t \bar{\zeta}_t dt - 2\psi_t^{-1} \mathrm{Re}[D_z \psi_t \zeta_t] a_t D_{\bar{z}}^* \psi_t dt.$$

(Pay attention that the dB_t terms cancel.)

Then,

$$\begin{aligned} d|\zeta_t|^2 &= -2\Lambda \langle \bar{\zeta}_t, a_t \zeta_t \rangle dt + 2\mathrm{Re}\big[\langle \bar{\zeta}_t, a_t D_{\bar{z},z}^* \psi_t \zeta_t \rangle + \langle \bar{\zeta}_t, a_t D_{\bar{z},\bar{z}}^* \psi_t \bar{\zeta}_t \rangle\big] dt \\ &\quad - 4\psi_t^{-1} \mathrm{Re}[D_z \psi_t \zeta_t] \mathrm{Re}[D_z \psi_t a_t \zeta_t] dt. \end{aligned}$$

Have in mind that $\psi_t \geq \epsilon$ for $t \in [\mathfrak{s}, \mathfrak{t}]$. Then, by Young's inequality, we can find some constant $C(\epsilon, \psi)$ depending on ϵ and ψ only[22], such that

$$d|\zeta_t|^2 \leq \big[C(\epsilon, \psi) - 2\Lambda\big] \langle \bar{\zeta}_t, a_t \zeta_t \rangle dt + \epsilon^2 |\zeta_t|^2 dt, \quad \mathfrak{s} \leq t \leq \mathfrak{t}. \tag{8.54}$$

Consider now some real R such that $R^2 \geq 2 \sup_{z \in \mathcal{D}}[|z|^2]$. Then, by Lemma 6.8,

$$d\bigg[\big[(R^2 - |Z_t|^2)\psi_t^{-1}\big] \exp\bigg(\int_0^t L\psi_r dr\bigg)\bigg] = -\exp\bigg(\int_0^t L\psi_r dr\bigg) dt + dm_t,$$

where $(m_t)_{t\geq 0}$ stands for a generic martingale term whose value may vary from line to line. In particular, for a small real $\delta > 0$,

$$\begin{aligned} &d\bigg[\big[(R^2 - |Z_t|^2)\psi_t^{-1}\big] \exp\bigg(\int_0^t (1-\delta) L\psi_r dr\bigg)\bigg] \\ &= \big[-\delta(R^2 - |Z_t|^2)\psi_t^{-1} L\psi_t - 1\big] \exp\bigg(\int_0^t (1-\delta) L\psi_r dr\bigg) dt + dm_t, \quad t \geq 0. \end{aligned} \tag{8.55}$$

Finally, from (8.54) and (8.55),

$$\begin{aligned} &d\bigg[\big[(R^2 - |Z_t|^2)\psi_t^{-1} |\zeta_t|^2\big] \exp\bigg(\int_0^t (1-\delta) L\psi_r dr\bigg)\bigg] \\ &\leq \big[\big(\epsilon^2 - \delta L\psi_t\big)(R^2 - |Z_t|^2)\psi_t^{-1} - 1\big] |\zeta_t|^2 \exp\bigg(\int_0^t (1-\delta) L\psi_r dr\bigg) dt \\ &\quad + \big(C(\epsilon,\psi) - 2\Lambda\big) \big[(R^2 - |Z_t|^2)\psi_t^{-1}\big] \langle \bar{\zeta}_t, a_t \zeta_t \rangle \exp\bigg(\int_0^t (1-\delta) L\psi_r dr\bigg) dt \\ &\quad + dm_t, \end{aligned} \tag{8.56}$$

[22] We here specify the dependence on ψ since ψ may vary in the statement of Proposition 8.9.

for $\mathfrak{s} \leq t \leq \mathfrak{t}$. Choose ϵ small enough such that $\epsilon R^2 \leq 1/2$ and then δ small enough such that

$$\delta^{-1} \geq 2R^2\epsilon^{-1} \sup\big\{-\mathrm{Trace}(aD^2_{z,\bar{z}}\psi(z)),\ z \in \mathcal{D},\ a \in \mathcal{H}_d : \mathrm{Trace}(a) = 1\big\}, \tag{8.57}$$

so that

$$\delta R^2\epsilon^{-1} \sup\big\{-\mathrm{Trace}(aD^2_{z,\bar{z}}\psi(z)),\ z \in \mathcal{D},\ a \in \mathcal{H}_d : \mathrm{Trace}(a) = 1\big\} \leq \frac{1}{2}.$$

Then, for any $\mathfrak{s} \leq t \leq \mathfrak{t}$,

$$\big(\epsilon^2 - \delta L\psi_t\big)(R^2 - |Z_t|^2)\psi_t^{-1} - 1 \leq \big(\epsilon^2 - \delta L\psi_t\big)R^2\epsilon^{-1} - 1 \leq 0,$$

so that the first term in the RHS in (8.56) is non-positive. Choose finally $\Lambda = C(\epsilon, \psi)/2$ to cancel the second term in the RHS in (8.56). Then,

$$d\bigg[\big[(R^2 - |Z_t|^2)\psi_t^{-1}|\zeta_t|^2\big] \exp\bigg(\int_0^t (1-\delta)L\psi_r dr\bigg)\bigg] \leq dm_t, \quad \mathfrak{s} \leq t \leq \mathfrak{t}.$$

Finally,

Proposition 8.10. *Let ψ be a plurisuperharmonic function describing the domain $\mathcal{D}$ as in (**A**). Then, there exists a positive real $\epsilon_3 > 0$ such that for any $0 < \epsilon < \epsilon_3$, we can find a constant $C(\epsilon, \psi)$, depending on ϵ and ψ only, such that, for any stopping time $\mathfrak{s}$ at which $\psi(Z^s_{\mathfrak{s}}) > \epsilon$, for $\Lambda = C(\epsilon, \psi)/2$ in (8.53) and $R^2 \geq 2\sup_{z\in\mathcal{D}}[|z|^2]$, the* derivative quantity *obtained by perturbing the time speed as in (8.52) and the measure as in (8.53)*

$$\bar{\Gamma}^{(3)}_t = (R^2 - |Z^s_t|^2)\psi^{-1}(Z^s_t)|\zeta_t|^2, \quad t \geq \mathfrak{s},$$

satisfies up to time $\mathfrak{t} = \inf\{t \geq \mathfrak{s} : \psi(Z^s_t) \leq \epsilon\}$ (provided that $(\hat{Z}^{s+\varepsilon}_t)_{0\leq t\leq\mathfrak{t}}$ is well differentiable w.r.t. ε)

$$\begin{aligned}
&\mathbb{E}\bigg[\exp\bigg(\int_0^{t\wedge\mathfrak{t}} (1-\delta)\mathrm{Trace}[a_r D^2_{z,\bar{z}}\psi(Z^s_r)]dr\bigg)\bar{\Gamma}^{(3)}_{t\wedge\mathfrak{t}}|\mathcal{F}_{\mathfrak{s}}\bigg] \\
&\leq \exp\bigg(\int_0^{\mathfrak{s}} (1-\delta)\mathrm{Trace}[a_r D^2_{z,\bar{z}}\psi(Z^s_r)]dr\bigg)\bar{\Gamma}^{(3)}_{\mathfrak{s}}, \quad t \geq \mathfrak{s},
\end{aligned}$$

with δ as in (8.57).

8.7. **Interpolation between the Interior and the Boundary.** It now remains to gather the estimates at and away the boundary. To do, we introduce an interpolated version of the *derivative quantity*.

The idea is the same as in the previous subsection: we couple at the same time several perturbations. Specifically, we here make use of the three possible types of perturbations discussed in Subsections 8.1, 8.2 and 8.4: the control perturbation is given by (8.11) and (8.35), i.e.

$$\begin{aligned}
&\frac{d}{d\varepsilon}\big[P(Z^s_t, \hat{Z}^{s+\varepsilon}_t - Z^s_t)\big] \\
&= |D_z\psi_t|^{-2}\big[D^2_{\bar{z},z}\psi_t\zeta_t D_z\psi_t + D^2_{\bar{z},\bar{z}}\psi_t\bar{\zeta}_t D_z\psi_t \\
&\qquad - D^*_{\bar{z}}\psi_t(D^2_{z,\bar{z}}\psi_t\bar{\zeta}_t)^* - D^*_{\bar{z}}\psi_t(D^2_{z,z}\psi_t\zeta_t)^*\big], \\
&:= Q_t\zeta_t,
\end{aligned} \tag{8.58}$$

the time-change perturbation is given by a variation of (8.20), namely

$$\frac{d}{d\varepsilon}\big[T(Z_t^s,\hat{Z}_t^{s+\varepsilon}-Z_t^s)\big]_{|\varepsilon=0}=(\lambda-1)\psi^{-1}(Z_t^s)\mathrm{Re}\big[D_z\psi(Z_t^s)\zeta_t\big], \tag{8.59}$$

for some real $\lambda\in(0,1)$ to be chosen later on, and the measure perturbation is given as a variation of (8.28):

$$\begin{aligned}&\frac{d}{d\varepsilon}\big[G(Z_t^s,\hat{Z}_t^{s+\varepsilon}-Z_t^s)\big]_{|\varepsilon=0}\\&\quad=(-2\lambda+\lambda^2+2)\psi^{-3/2}(Z_t^s)\mathrm{Re}\big[D_z\psi(Z_t^s)\zeta_t\big]\bar{\sigma}_t^* D_{\bar{z}}\psi(Z_t^s).\end{aligned} \tag{8.60}$$

(We here say a variation of (8.28) since the perturbation now involves $(\bar{\zeta}_t)_{\mathfrak{s}\le t\le\mathfrak{t}}$ as well. Obviously, this doesn't change the global strategy.) The dynamics of $(\hat{Z}_t^{s+\varepsilon})_{\mathfrak{s}\le t\le\mathfrak{t}}$ then read (compare with (8.10), (8.18) and (8.25))

$$\begin{aligned}d\hat{Z}_t^{s+\varepsilon}&=\psi^{1/2}(\hat{Z}_t^{s+\varepsilon})T(Z_t^s,\hat{Z}_t^{s+\varepsilon}-Z_t^s)\exp\big(P(Z_t^s,\hat{Z}_t^{s+\varepsilon}-Z_t^s)\big)\\&\qquad\times\sigma_t\big[dB_t+G(Z_t^s,\hat{Z}_t^{s+\varepsilon}-Z_t^s)dt\big]\\&\quad+T^2(Z_t^s,\hat{Z}_t^{s+\varepsilon}-Z_t^s)\exp\big(P(Z_t^s,\hat{Z}_t^{s+\varepsilon}-Z_t^s)\big)\\&\qquad\times a_t\exp\big(-P(Z_t^s,\hat{Z}_t^{s+\varepsilon}-Z_t^s)\big)D_{\bar{z}}^*\psi(Z_t^{s+\varepsilon})dt,\end{aligned}$$

for $\mathfrak{s}\le t\le\mathfrak{t}$.

Following the localization procedure described in the statement of Propositions 8.2, 8.4 and 8.7, the time indices t we consider in this subsection are always assumed to belong to the interval $[\mathfrak{s},\mathfrak{t}]$, where $\mathfrak{s}$ is some stopping time at which $\epsilon'<\psi(Z_t^s)<\epsilon$, for an additional positive real ϵ' [23] and $\mathfrak{t}=\inf\{t>\mathfrak{s}:\psi(Z_t^s)\notin]\epsilon',\epsilon[\}$. In particular for $t\in[\mathfrak{s},\mathfrak{t}]$, $\psi(Z_t^s)$ belongs to $[\epsilon',\epsilon]$.

We also make use of the same abbreviated notation as above: we get rid of the symbol hat "$\hat{}$" and of the superscript s for more simplicity in $(\hat{\zeta}_t^s)_{\mathfrak{s}\le t\le\mathfrak{t}}$; we also write ψ_t for $\psi(Z_t^s)$ and $L\psi_t$ for $\mathrm{Trace}[a_tD_{z,\bar{z}}^2\psi(Z_t^s)]$, $\mathfrak{s}\le t\le\mathfrak{t}$.

Then, we can differentiate the dynamics of $(\hat{Z}_t^{s+\varepsilon})_{t\ge0}$ according to the rules prescribed above. Following (8.36), we obtain

$$\begin{aligned}&d\zeta_t\\&=\big[\lambda\psi_t^{-1/2}\mathrm{Re}\big[D_z\psi_t\zeta_t\big]+\psi_t^{1/2}Q_t\zeta_t\big]\sigma_tdB_t+\psi_t^{1/2}\Xi_ta_tD_{\bar{z}}^*\psi_tdt\\&\quad+\big(N^{-1}\mathcal{E}_t+\mathcal{E}_t^{1/2}\big)|\zeta_t|r_tdt+2(\lambda-1)\psi_t^{-1}\mathrm{Re}\big[D_z\psi_t\zeta_t\big]a_tD_{\bar{z}}^*\psi_tdt,\quad t\ge0,\end{aligned}$$

where $(r_t)_{\mathfrak{s}\le t\le\mathfrak{t}}$ stands for a generic process, bounded by some constant C depending on (**A**) only. (Here and only here $(r_t)_{\mathfrak{s}\le t\le\mathfrak{t}}$ has values in $\mathbb{C}^d$. Below, it has values in $\mathbb{C}$.) Above, $\mathcal{E}_t:=D_z\psi_ta_tD_{\bar{z}}^*\psi_t$ and N denotes a real greater than 1 such that $\psi=N\psi^0$ where ψ^0 is some reference choice of the plurisuperharmonic function describing $\mathcal{D}$, such that $\mathrm{Trace}[aD_{z,\bar{z}}^2\psi^0(z)]\le-1$ for any $z\in\mathcal{D}$ and any positive Hermitian matrix a of trace 1.

Now, $N^{-1}\mathcal{E}_t$ is bounded by $C\mathcal{E}_t^{1/2}$, $\mathfrak{s}\le t\le\mathfrak{t}$, up to a modificiation of C. (Pay attention that C is independent of N.) Therefore, using the boundedness of $|Q_t\zeta_t|/|\zeta_t|$ (see (8.35)),

[23] The values of both ϵ and ϵ' will be specified later on.

$\mathfrak{s} \leq t \leq \mathfrak{t}$,

$$
\begin{aligned}
& d|\zeta_t|^2 \\
& = \lambda \psi_t^{-1/2} \mathrm{Re}\big[D_z \psi_t \zeta_t\big] \big(\langle \bar{\zeta}_t, \sigma_t dB_t \rangle + \langle \zeta_t, \bar{\sigma}_t d\bar{B}_t \rangle \big) \\
& \quad + \psi_t^{1/2} \big(\langle \bar{\zeta}_t, Q_t \zeta_t \sigma_t dB_t \rangle + \langle \zeta_t, \overline{Q_t \zeta_t} \bar{\sigma}_t d\bar{B}_t \rangle \big) \\
& \quad + 4(\lambda - 1) \psi_t^{-1} \mathrm{Re}\big[D_z \psi_t \zeta_t\big] \mathrm{Re}\big[D_z \psi_t a_t \zeta_t\big] dt + 2 \psi_t^{1/2} \Xi_t \mathrm{Re}\big[D_z \psi_t a_t \zeta_t\big] dt \\
& \quad + \big[\lambda^2 \psi_t^{-1} \mathrm{Re}^2[D_z \psi_t \zeta_t] + \lambda N |\zeta_t|^2 r_t + \psi_t |\zeta_t|^2 r_t + \mathcal{E}_t^{1/2} |\zeta_t|^2 r_t\big] dt.
\end{aligned}
\tag{8.61}
$$

Now, from (8.42),

$$
\begin{aligned}
d\psi_t^{-\lambda} = & -\lambda \psi_t^{-\lambda - 1/2} \big[D_z \psi_t \sigma_t dB_t + D_{\bar{z}} \psi_t \bar{\sigma}_t d\bar{B}_t\big] \\
& \qquad - \lambda(1-\lambda) \psi_t^{-(1+\lambda)} D_z \psi_t a_t D_{\bar{z}}^* \psi_t dt - \lambda \psi_t^{-\lambda} L \psi_t dt.
\end{aligned}
\tag{8.62}
$$

By (8.43), for $K \geq 1$ to be chosen later on,

$$
\begin{aligned}
& d\big[\exp(-K\psi_t)\big] \\
& = -K \exp(-K\psi_t) \psi_t^{1/2} \big[D_z \psi_t \sigma_t dB_t + D_{\bar{z}} \psi_t \bar{\sigma}_t d\bar{B}_t\big] \\
& \quad + [K^2 \psi_t - 2K] \exp(-K\psi_t) D_z \psi_t a_t D_{\bar{z}}^* \psi_t dt - K \exp(-K\psi_t) \psi_t L \psi_t dt,
\end{aligned}
$$

so that

$$
\begin{aligned}
& d\big[\exp(-K\psi_t) \psi_t^{-\lambda}\big] \\
& = -\big[\lambda \psi_t^{-1/2} + K \psi_t^{1/2}\big] \exp(-K\psi_t) \psi_t^{-\lambda} \big[D_z \psi_t \sigma_t dB_t + D_{\bar{z}} \psi_t \bar{\sigma}_t d\bar{B}_t\big] \\
& \quad + \big[K^2 \psi_t + 2\lambda K - 2K - \lambda(1-\lambda) \psi_t^{-1}\big] \exp(-K\psi_t) \psi_t^{-\lambda} D_z \psi_t a_t D_{\bar{z}}^* \psi_t dt \\
& \quad - \big[\lambda + K\psi_t\big] \exp(-K\psi_t) \psi_t^{-\lambda} L \psi_t dt.
\end{aligned}
$$

Then, by (8.61) and the above equality,

$$
\begin{aligned}
& d\big[\exp(-K\psi_t) \psi_t^{-\lambda} |\zeta_t|^2\big] \\
& = (4\lambda - 2\lambda^2 - 4 - 2\lambda K \psi_t) \exp(-K\psi_t) \psi_t^{-(1+\lambda)} \mathrm{Re}\big[D_z \psi_t \zeta_t\big] \mathrm{Re}\big[D_z \psi_t a_t \zeta_t\big] dt \\
& \quad + 2\Xi_t \exp(-K\psi_t) \psi_t^{1/2 - \lambda} \mathrm{Re}\big[D_z \psi_t a_t \zeta_t\big] dt \\
& \quad + \exp(-K\psi_t) \psi_t^{-\lambda} \big[\lambda^2 \psi_t^{-1} \mathrm{Re}^2[D_z \psi_t \zeta_t] + \lambda N |\zeta_t|^2 r_t \\
& \qquad\qquad\qquad + N K \psi_t |\zeta_t|^2 r_t + \mathcal{E}_t^{1/2} |\zeta_t|^2 r_t\big] dt \\
& \quad + \big[K^2 \psi_t + 2\lambda K - 2K - \lambda(1-\lambda) \psi_t^{-1}\big] \exp(-K\psi_t) \psi_t^{-\lambda} \mathcal{E}_t |\zeta_t|^2 dt \\
& \quad - \big[\lambda + K\psi_t\big] \exp(-K\psi_t) \psi_t^{-\lambda} L \psi_t |\zeta_t|^2 dt + dm_t,
\end{aligned}
\tag{8.63}
$$

where $(m_t)_{\mathfrak{s} \leq t \leq \mathfrak{s}}$ stands for a generic martingale term.

By the specific choice we made for $(\Xi_t)_{\mathfrak{s}\le t\le \mathfrak{t}}$, see (8.60), and by Young's inequality,

$$
\begin{aligned}
&d\big[\exp(-K\psi_t)\psi_t^{-\lambda}|\zeta_t|^2\big]\\
&\le \lambda^2\psi_t^{-(1+\lambda)}\mathrm{Re}^2[D_z\psi_t\zeta_t]dt\\
&\quad + C\big(\lambda K N^2 + \lambda N + K^{-1} + NK\psi_t\big)\exp(-K\psi_t)\psi_t^{-\lambda}|\zeta_t|^2 dt\\
&\quad + \big[K^2\psi_t - (1-2\lambda)K\big]\exp(-K\psi_t)\psi_t^{-\lambda}\mathcal{E}_t|\zeta_t|^2 dt\\
&\quad - \big[\lambda + K\psi_t\big]\exp(-K\psi_t)\psi_t^{-\lambda}L\psi_t|\zeta_t|^2 dt + dm_t.
\end{aligned}
\tag{8.64}
$$

Replacing $-\lambda$ by $(1-\lambda)/2$ in (8.62), we obtain in a similar way

$$
\begin{aligned}
&d\psi_t^{(1-\lambda)/2}\\
&= \frac{1-\lambda}{2}\psi_t^{-\lambda/2}\big[D_z\psi_t\sigma_t dB_t + D_{\bar z}\psi_t\bar\sigma_t d\bar B_t\big]\\
&\quad + \frac{(1-\lambda)(3-\lambda)}{4}\psi_t^{-(1+\lambda)/2}D_z\psi_t a_t D_{\bar z}^*\psi_t dt + \frac{1-\lambda}{2}\psi_t^{(1-\lambda)/2}L\psi_t dt.
\end{aligned}
\tag{8.65}
$$

Below, we make use of (8.65) but at point $s+\varepsilon$ instead of ε. We obtain

$$
\begin{aligned}
&d\big[\psi^{(1-\lambda)/2}(\hat Z_t^{s+\varepsilon})\big]\\
&= \frac{1-\lambda}{2}\psi^{-\lambda/2}(\hat Z_t^{s+\varepsilon})T(Z_t^s, \hat Z_t^{s+\varepsilon} - Z_t^s)\\
&\quad \times \big[D_z\psi(\hat Z_t^{s+\varepsilon})\exp\big(P(Z_t^s, \hat Z_t^{s+\varepsilon} - Z_t^s)\big)\sigma_t\big(dB_t + G(Z_t^s, \hat Z_t^{s+\varepsilon} - Z_t^s)dt\big)\\
&\quad\ + D_{\bar z}\psi(\hat Z_t^{s+\varepsilon})\exp\big(\bar P(Z_t^s, \hat Z_t^{s+\varepsilon} - Z_t^s)\big)\bar\sigma_t\big(d\bar B_t + \bar G(Z_t^s, \hat Z_t^{s+\varepsilon} - Z_t^s)dt\big)\big]\\
&\quad + \frac{(1-\lambda)(3-\lambda)}{4}\psi^{-(1+\lambda)/2}(\hat Z_t^{s+\varepsilon})T^2(Z_t^s, \hat Z_t^{s+\varepsilon} - Z_t^s)D_z\psi(\hat Z_t^{s+\varepsilon})\\
&\quad \times \exp\big(P(Z_t^s, \hat Z_t^{s+\varepsilon} - Z_t^s)\big)a_t\exp\big(-P(Z_t^s, \hat Z_t^{s+\varepsilon} - Z_t^s)\big)D_{\bar z}^*\psi(\hat Z_t^{s+\varepsilon})dt\\
&\quad + \frac{1-\lambda}{2}\psi^{(1-\lambda)/2}(\hat Z_t^{s+\varepsilon})T^2(Z_t^s, \hat Z_t^{s+\varepsilon} - Z_t^s)\\
&\quad \times \mathrm{Trace}\big[\exp\big(P(Z_t^s, \hat Z_t^{s+\varepsilon} - Z_t^s)\big)\\
&\qquad\quad \times a_t\exp\big(-P(Z_t^s, \hat Z_t^{s+\varepsilon} - Z_t^s)\big)D_{z,\bar z}^2\psi(\hat Z_t^{s+\varepsilon})\big]dt.
\end{aligned}
\tag{8.66}
$$

We now differentiate according to the rules prescribed above (see in particular (8.58), (8.59) and (8.60)). Using (8.46), we obtain

$$
\begin{aligned}
&(1-\lambda)d\big[\psi_t^{-(1+\lambda)/2}\mathrm{Re}[D_z\psi_t\zeta_t]\big] \\
&= \frac{1-\lambda}{2}\psi_t^{-\lambda/2}\big[-\psi_t^{-1}\mathrm{Re}[D_z\psi_t\zeta_t]+r_t|\zeta_t|\big]\big[D_z\psi_t\sigma_t dB_t + D_{\bar z}\psi_t\bar\sigma_t d\bar B_t\big] \\
&\quad + (1-\lambda)\psi_t^{-\lambda/2}\Xi_t D_z\psi_t a_t D_{\bar z}^*\psi_t dt \\
&\quad + \frac{(1-\lambda)(3-\lambda)}{4}\big[-1-\lambda-2+2\lambda\big]\psi_t^{-(3+\lambda)/2}\mathrm{Re}[D_z\psi_t\zeta_t]D_z\psi_t a_t D_{\bar z}^*\psi_t dt \\
&\quad + \frac{(1-\lambda)(3-\lambda)}{4}\psi_t^{-(1+\lambda)/2}r_t|\zeta_t|D_z\psi_t a_t D_{\bar z}^*\psi_t dt \\
&\quad + \frac{1-\lambda}{2}\big[1-\lambda-2+2\lambda\big]\psi_t^{-(1+\lambda)/2}\mathrm{Re}[D_z\psi_t\zeta_t]L\psi_t dt \\
&\quad + (1-\lambda)\psi_t^{(1-\lambda)/2}\big[\mathrm{Re}\big(D_zL\psi_t\zeta_t\big)+\mathrm{Re}\big(\mathrm{Trace}[Q_t\zeta_t a_t D_{z,\bar z}^2\psi_t]\big)\big]dt.
\end{aligned}
$$

In a shorter way,

$$
\begin{aligned}
&d\big[\psi_t^{-(1+\lambda)/2}\mathrm{Re}[D_z\psi_t\zeta_t]\big] \\
&= \frac{1}{2}\psi_t^{-\lambda/2}\big[-\psi_t^{-1}\mathrm{Re}[D_z\psi_t\zeta_t]+r_t|\zeta_t|\big]\big[D_z\psi_t\sigma_t dB_t + D_{\bar z}\psi_t\bar\sigma_t d\bar B_t\big] \\
&\quad + \frac{3-\lambda}{4}(\lambda-3)\psi_t^{-(3+\lambda)/2}\mathrm{Re}[D_z\psi_t\zeta_t]D_z\psi_t a_t D_{\bar z}^*\psi_t dt \\
&\quad + \frac{3-\lambda}{4}\psi_t^{-(1+\lambda)/2}r_t|\zeta_t|D_z\psi_t a_t D_{\bar z}^*\psi_t dt + \psi_t^{-\lambda/2}\Xi_t D_z\psi_t a_t D_{\bar z}^*\psi_t dt \\
&\quad + \frac{1}{2}(\lambda-1)\psi_t^{-(1+\lambda)/2}\mathrm{Re}[D_z\psi_t\zeta_t]L\psi_t dt \\
&\quad + \psi_t^{(1-\lambda)/2}\big[\mathrm{Re}\big(D_zL\psi_t\zeta_t\big)+\mathrm{Re}\big(\mathrm{Trace}[Q_t\zeta_t a_t D_{z,\bar z}^2\psi_t]\big)\big]dt.
\end{aligned}
$$

Finally, taking the square, we obtain

$$
\begin{aligned}
&d\big[\psi_t^{-(1+\lambda)}\mathrm{Re}^2[D_z\psi_t\zeta_t]\big] \\
&= \bigg\{\frac{1}{2}\psi_t^{-(2+\lambda)}\mathrm{Re}^2[D_z\psi_t\zeta_t] + \psi_t^{-\lambda}r_t|\zeta_t|^2 + \psi_t^{-(1+\lambda)}\mathrm{Re}[D_z\psi_t\zeta_t]r_t|\zeta_t| \\
&\quad + \frac{(3-\lambda)}{2}(\lambda-3)\psi_t^{-(2+\lambda)}\mathrm{Re}^2[D_z\psi_t\zeta_t] + 2\psi_t^{-(1/2+\lambda)}\mathrm{Re}[D_z\psi_t\zeta_t]\Xi_t\bigg\} \\
&\qquad\quad \times D_z\psi_t a_t D_{\bar z}^*\psi_t dt \\
&\quad + (\lambda-1)\psi_t^{-(1+\lambda)}\mathrm{Re}^2[D_z\psi_t\zeta_t]L\psi_t dt \\
&\quad + 2\psi_t^{-\lambda}\mathrm{Re}[D_z\psi_t\zeta_t]\big[\mathrm{Re}\big(D_zL\psi_t\zeta_t\big)+\mathrm{Re}\big(\mathrm{Trace}[Q_t\zeta_t a_t D_{z,\bar z}^2\psi_t]\big)\big]dt + dm_t.
\end{aligned}
$$

In abbreviated notations, we deduce

$$
\begin{aligned}
& d\big[\psi_t^{-(1+\lambda)}\mathrm{Re}^2[D_z\psi_t\zeta_t]\big] \\
& = \frac{1+(3-\lambda)(\lambda-3)}{2}\psi_t^{-(2+\lambda)}\mathrm{Re}^2[D_z\psi_t\zeta_t]\mathcal{E}_t dt \\
& \quad + 2\psi_t^{-(1/2+\lambda)}\mathrm{Re}[D_z\psi_t\zeta_t]\Xi_t\mathcal{E}_t dt \\
& \quad + \psi_t^{-(1+\lambda)}\mathrm{Re}[D_z\psi_t\zeta_t]\mathcal{E}_t|\zeta_t|r_t dt \\
& \quad + (\lambda-1)\psi_t^{-(1+\lambda)}\mathrm{Re}^2[D_z\psi_t\zeta_t]L\psi_t dt + N^2\psi_t^{-\lambda}|\zeta_t|^2 r_t dt + dm_t.
\end{aligned}
$$

Recall now from (8.60) that $\Xi_t = (-2\lambda+\lambda^2+2)\psi_t^{-3/2}\mathrm{Re}[D_z\psi_t\zeta_t]$. Then, applying Young's inequality to the second term in the above RHS,

$$
\begin{aligned}
& d\big[\psi_t^{-(1+\lambda)}\mathrm{Re}^2[D_z\psi_t\zeta_t]\big] \\
& \leq \big(-\frac{1}{2}\lambda+\frac{3}{2}\lambda^2\big)\psi_t^{-(2+\lambda)}\mathcal{E}_t\mathrm{Re}^2[D_z\psi_t\zeta_t]dt \\
& \quad + C(\lambda^{-1}\mathcal{E}_t+N^2)\psi_t^{-\lambda}|\zeta_t|^2 dt + (\lambda-1)\psi_t^{-(1+\lambda)}\mathrm{Re}^2[D_z\psi_t\zeta_t]L\psi_t dt + dm_t.
\end{aligned}
\tag{8.67}
$$

Choose now $\epsilon \leq 1$ and $\lambda \leq \epsilon$ small enough such that $-\lambda/2+3\lambda^2/2<0$ and $N=K=\epsilon^{-1/4}$. Then, (8.64) writes for $\psi_t \leq \epsilon$

$$
\begin{aligned}
& d\big[\exp(-K\psi_t)\psi_t^{-\lambda}|\zeta_t|^2\big] \\
& \leq \lambda^2\psi_t^{-(1+\lambda)}\mathrm{Re}^2[D_z\psi_t\zeta_t]dt + C\epsilon^{1/4}\exp(-K\psi_t)\psi_t^{-\lambda}|\zeta_t|^2 dt \\
& \quad + \big[3\epsilon^{1/2}-\epsilon^{-1/4}\big]\exp(-K\psi_t)\psi_t^{-\lambda}\mathcal{E}_t|\zeta_t|^2 dt + dm_t.
\end{aligned}
\tag{8.68}
$$

In the same way, (8.67) has the form

$$
\begin{aligned}
d\big[\psi_t^{-(1+\lambda)}\mathrm{Re}^2[D_z\psi_t\zeta_t]\big] & \leq C(\lambda^{-1}\mathcal{E}_t+\epsilon^{-1/2})\exp(-K\psi_t)\psi_t^{-\lambda}|\zeta_t|^2 dt \\
& \quad + (\lambda-1)\psi_t^{-(1+\lambda)}\mathrm{Re}^2[D_z\psi_t\zeta_t]L\psi_t dt + dm_t.
\end{aligned}
\tag{8.69}
$$

Consider now the *modified derivative quantity*

$$
\bar{\Gamma}_t = \exp(-K\psi_t)\psi_t^{-\lambda}|\zeta_t|^2 + 2\lambda\epsilon^{1/4}\psi_t^{-(1+\lambda)}\mathrm{Re}^2[D_z\psi_t\zeta_t].
$$

From (8.68) and (8.69), we obtain

$$
\begin{aligned}
d\bar{\Gamma}_t & \leq C\epsilon^{1/4}\exp(-K\psi_t)\psi_t^{-\lambda}|\zeta_t|^2 dt \\
& \quad + \big(C\epsilon^{1/4}-\epsilon^{-1/4}\big)\exp(-K\psi_t)\psi_t^{-\lambda}\mathcal{E}_t|\zeta_t|^2 dt \\
& \quad + \big[2\lambda(\lambda-1)\epsilon^{1/4}L\psi_t+\lambda^2\big]\psi_t^{-(1+\lambda)}\mathrm{Re}^2[D_z\psi_t\zeta_t]dt + dm_t.
\end{aligned}
$$

For $C\epsilon^{1/4}-\epsilon^{-1/4}<0$, we deduce

$$
\begin{aligned}
d\bar{\Gamma}_t & \leq C\epsilon^{1/4}\exp(-K\psi_t)\psi_t^{-\lambda}|\zeta_t|^2 dt \\
& \quad + \big[\lambda^2(2L\psi_t^0+1)-2\lambda\epsilon^{1/4}L\psi_t\big]\psi_t^{-(1+\lambda)}\mathrm{Re}^2[D_z\psi_t\zeta_t]dt + dm_t.
\end{aligned}
$$

Since $L\psi_t^0 \leq -1$, we finally deduce

$$
\begin{aligned}
d\bar{\Gamma}_t &\leq C\epsilon^{1/4}\exp(-K\psi_t)\psi_t^{-\lambda}|\zeta_t|^2 dt \\
&\quad + \big[\lambda^2 L\psi_t^0 - 2\lambda\epsilon^{1/4}L\psi_t\big]\psi_t^{-(1+\lambda)}\mathrm{Re}^2[D_z\psi_t\zeta_t]dt + dm_t \\
&\leq C\epsilon^{1/4}\exp(-K\psi_t)\psi_t^{-\lambda}|\zeta_t|^2 dt \\
&\quad + 2\big[(\lambda/2-1)L\psi_t\big]\lambda\epsilon^{1/4}\psi_t^{-(1+\lambda)}\mathrm{Re}^2[D_z\psi_t\zeta_t]dt + dm_t.
\end{aligned}
$$

Following (8.50) and (8.51), we deduce that

$$
d\bigg[\exp\bigg(\int_0^t (1-\lambda/2)L\psi_r dr\bigg)\bar{\Gamma}_t\bigg] \leq dm_t, \quad \mathfrak{s} \leq t \leq \mathfrak{t}, \tag{8.70}
$$

for ϵ small enough and $\lambda \leq \epsilon$.

We deduce

Proposition 8.11. *Let ψ be a plurisuperharmonic function describing the domain $\mathcal{D}$ as in (**A**). Then, there exists a positive real $\epsilon_2 > 0$ such that for any $0 < \epsilon' < \epsilon < \epsilon_2$ and $0 < \lambda < \epsilon$, for $N = K = \epsilon^{-1/4}$, $\psi = N\psi^0$ (with ψ^0 as in the statement of Proposition 8.9) and any stopping time $\mathfrak{s}$ at which $\psi(Z^s_{\mathfrak{s}}) \in [\epsilon', \epsilon]$, the* derivative quantity *obtained by perturbing the control parameter as in* (8.58), *the time speed as in* (8.59) *and the measure as in* (8.60)*:*

$$
\bar{\Gamma}^{(2)}_t = \exp(-K\psi_t)\psi^{-\lambda}(Z^s_t)|\zeta_t|^2 + 2\lambda\epsilon^{1/4}\psi_t^{-(1+\lambda)}\mathrm{Re}^2[D_z\psi(Z^s_t)\zeta_t], \quad t \geq \mathfrak{s},
$$

satisfies up to time $\mathfrak{t} = \inf\{t \geq \mathfrak{s} : \psi(Z^s_t) \notin]\epsilon', \epsilon[\}$ (provided that $(\hat{Z}^{s+\varepsilon}_t)_{\mathfrak{s}\leq t\leq \mathfrak{t}}$ is well differentiable w.r.t. ε)

$$
\begin{aligned}
&\mathbb{E}\bigg[\exp\bigg(\int_0^{t\wedge\mathfrak{t}} (1-\delta)\mathrm{Trace}[a_r D^2_{z,\bar{z}}\psi(Z^s_r)]dr\bigg)\bar{\Gamma}^{(2)}_{t\wedge\mathfrak{t}}|\mathcal{F}_{\mathfrak{s}}\bigg] \\
&\leq \exp\bigg(\int_0^{\mathfrak{s}} (1-\delta)\mathrm{Trace}[a_r D^2_{z,\bar{z}}\psi(Z^s_r)]dr\bigg)\bar{\Gamma}^{(2)}_{\mathfrak{s}}, \quad t \geq \mathfrak{s},
\end{aligned}
$$

with $\delta = \lambda/2$.

8.8. **Global Derivative Quantity.** The reader might understand the problem we are facing right now: above, we have defined three different *derivative quantities* according to the position of the underlying representation process in the domain $\mathcal{D}$. Surely, we must gather into a single one the three different parts to control the dynamics on the whole space.

Actually, the strategy is not so complicated. In what follows, we are given $0 < \epsilon < \min(\epsilon_1, \epsilon_2, \epsilon_3)$ in the statements of Propositions 8.9, 8.10 and 8.11 and we choose $\psi = \epsilon^{-1/4}\psi^0$ in each statement and $\lambda = \epsilon^2$ in the statement of Proposition 8.11. Then, the three different *derivative quantities* have the forms

$$
\begin{aligned}
\bar{\Gamma}^{(1)}_t &= \exp(-\epsilon^{-1/4}\psi_t)|\zeta_t|^2 + \psi_t^{-1}\mathrm{Re}^2\big[D_z\psi_t\zeta_t\big], \\
\bar{\Gamma}^{(2)}_t &= \exp(-\epsilon^{-1/4}\psi_t)\psi_t^{-\epsilon^2}|\zeta_t|^2 + 2\epsilon^{9/4}\psi_t^{-(1+\epsilon^2)}\mathrm{Re}^2\big[D_z\psi_t\zeta_t\big], \\
\bar{\Gamma}^{(3)}_t &= (R^2 - |Z_t|^2)\psi_t^{-1}|\zeta_t|^2.
\end{aligned} \tag{8.71}
$$

At this stage of the proof, the definitions of $\bar{\Gamma}^{(1)}$, $\bar{\Gamma}^{(2)}$ and $\bar{\Gamma}^{(3)}$ are purely formal since the perturbed process $(\hat{Z}^{s+\varepsilon}_t)_{t\geq 0}$ has not been defined in a global way yet. Obviously, $(Z_t)_{t\geq 0}$,

$(\psi_t)_{t\geq 0}$, $(\zeta_t)_{t\geq 0}$ and $(D_z\psi_t)_{t\geq 0}$ will be understood as $(Z_t^s)_{t\geq 0}$, solution of (8.9), $(\psi(Z_t^s))_{t\geq 0}$, $([d/d\varepsilon][\hat{Z}_t^{s+\varepsilon}])_{t\geq 0}$ and $(D_z\psi(Z_t^s))_{t\geq 0}$.

For the moment, we claim

Proposition 8.12. *Let $(Z_t)_{t\geq 0}$ be a process with values in $\mathcal{D}$ and $(\zeta_t)_{t\geq 0}$ be another process with values in $\mathbb{C}^d$. Setting $\psi_t = \psi(Z_t)$ and $D_z\psi_t = D_z\psi(Z_t)$, $t \geq 0$, consider $(\bar{\Gamma}_t^{(1)})_{t\geq 0}$, $(\bar{\Gamma}_t^{(2)})_{t\geq 0}$ and $(\bar{\Gamma}_t^{(3)})_{t\geq 0}$ as in (8.71).*

*Then, there exists a real $0 < \epsilon_0 < \min(\epsilon_1, \epsilon_2, \epsilon_3)$, depending on Assumption (**A**) only, such that for $\epsilon < \epsilon_0$, we can find three reals $\epsilon_4 < \epsilon/4$ and $\mu_2, \mu_3 > 0$, depending on ϵ and (**A**) only, such that*

$$
\begin{aligned}
&\psi_t = \epsilon \Rightarrow \mu_2\bar{\Gamma}_t^{(2)} \geq \mu_3\bar{\Gamma}_t^{(3)} \\
&\psi_t = \epsilon/2 \Rightarrow \bar{\Gamma}_t^{(1)} \geq \mu_2\bar{\Gamma}_t^{(2)} \left(+(1-2\epsilon^{9/4})\psi_t^{-1}\mathrm{Re}^2\left[D_z\psi_t\zeta_t\right]\right) \\
&\psi_t = \epsilon/4 \Rightarrow \mu_3\bar{\Gamma}_t^{(3)} \geq \mu_2\bar{\Gamma}_t^{(2)} \\
&\psi_t = \epsilon_4 \Rightarrow \mu_2\bar{\Gamma}_t^{(2)} \geq \bar{\Gamma}_t^{(1)} \left(+\left[\left(\frac{\epsilon}{2\epsilon_4}\right)^{\epsilon^2} - 1\right]|\zeta_t|^2\right).
\end{aligned}
$$

Above, additional terms in parentheses are positive for ϵ_0 small enough. They are useless in the whole Section 8. They will be useful in Section 9.

Proposition 8.12 may be understood through Figure 8.8 below. Each drawn curve stands for a possible graph of one of the three *derivative quantities* in Proposition 8.12. The boundary points of each curve (except the ones in 0 and ϵ) are bounded from below by the current point of another curve.

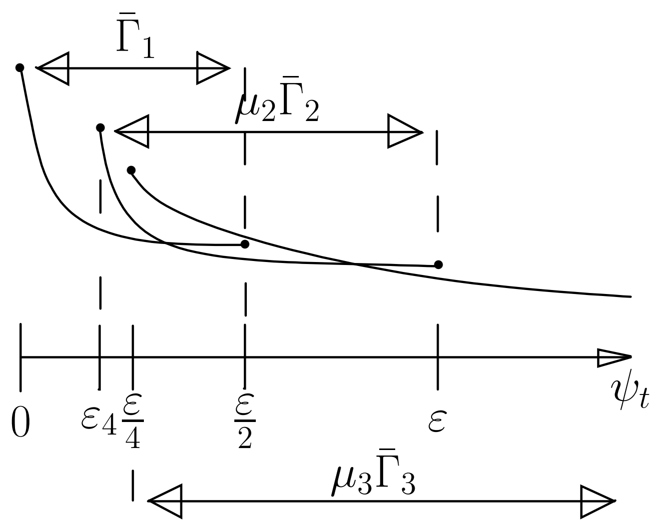

Figure 8.8. Representation of the *derivative quantities.*

Proof. When $\psi_t = \epsilon/2$, it is clear that

$$
\left(\frac{\epsilon}{2}\right)^{\epsilon^2}\bar{\Gamma}_t^{(2)} \leq \bar{\Gamma}_t^{(1)},
$$

provided $2\epsilon^{9/4} \leq 1$ (which is obviously true for ϵ small enough).

If $2(\epsilon/2)^{\epsilon^2}\epsilon^{9/4}\psi_t^{-\epsilon^2} = 1$ (i.e. $\psi_t = \epsilon_4$, with ϵ_4 much more smaller than $\epsilon/2$), then

$$\Big(\frac{\epsilon}{2}\Big)^{\epsilon^2}\bar{\Gamma}_t^{(2)} \geq \bar{\Gamma}_t^{(1)}.$$

We thus choose $\mu_2 = (\epsilon/2)^{\epsilon^2}$.

When $\psi_t = \epsilon$,

$$\epsilon^{1-\epsilon^2}R^{-2}\exp(-\epsilon^{3/4})\bar{\Gamma}_t^{(3)} \leq \bar{\Gamma}_t^{(2)}. \tag{8.72}$$

When $\psi_t = \vartheta\epsilon$,

$$\begin{aligned}\bar{\Gamma}_t^{(2)} &\leq (\vartheta\epsilon)^{1-\epsilon^2}\psi_t^{-1}|\zeta_t|^2 + 2\epsilon^{9/4-\epsilon^2}\vartheta^{-\epsilon^2}\|D_z\psi\|_\infty^2\psi_t^{-1}|\zeta_t|^2\\ &\leq (\vartheta\epsilon)^{1-\epsilon^2}\psi_t^{-1}|\zeta_t|^2 + 2\epsilon^{7/4-\epsilon^2}\vartheta^{-\epsilon^2}\|D_z\psi^0\|_\infty^2\psi_t^{-1}|\zeta_t|^2,\end{aligned}$$

since $\psi = \epsilon^{-1/4}\psi^0$.

Since $R^2 \geq 2\sup_{z\in\mathcal{D}}[|z|^2]$, we have $R^2 - \sup_{z\in\mathcal{D}}[|z|^2] \geq R^2/2$ so that $\bar{\Gamma}_t^{(3)} \geq (R^2/2)\psi_t^{-1}|\zeta_t|^2$. We deduce

$$\bar{\Gamma}_t^{(2)} \leq 2R^{-2}\big[(\vartheta\epsilon)^{1-\epsilon^2} + 2\epsilon^{7/4-\epsilon^2}\vartheta^{-\epsilon^2}\|D_z\psi^0\|_\infty^2\big]\bar{\Gamma}_t^{(3)}.$$

Finally,

$$\begin{aligned}\bar{\Gamma}_t^{(2)} &\leq 2R^{-2}\big[(\vartheta\epsilon)^{1-\epsilon^2} + 2\epsilon^{7/4-\epsilon^2}\vartheta^{-\epsilon^2}\|D_z\psi^0\|_\infty^2\big]\bar{\Gamma}_t^{(3)}\\ &\leq 2\exp(\epsilon^{3/4})\big[\vartheta^{1-\epsilon^2} + 2\epsilon^{3/4}\vartheta^{-\epsilon^2}\|D_z\psi^0\|_\infty^2\big]\epsilon^{1-\epsilon^2}R^{-2}\exp(-\epsilon^{3/4})\bar{\Gamma}_t^{(3)}.\end{aligned}$$

Choose $\vartheta = 1/4$. Then,

$$\bar{\Gamma}_t^{(2)} \leq 2\exp(\epsilon^{3/4})\big[4^{-1+\epsilon^2} + 2\cdot 4^{\epsilon^2}\epsilon^{3/4}\|D_z\psi^0\|_\infty^2\big]\epsilon^{1-\epsilon^2}R^{-2}\exp(-\epsilon^{3/4})\bar{\Gamma}_t^{(3)}.$$

Then, for ϵ small enough,

$$\bar{\Gamma}_t^{(2)} \leq \epsilon^{1-\epsilon^2}R^{-2}\exp(-\epsilon^{3/4})\bar{\Gamma}_t^{(3)}.$$

We finally choose $\mu_3 = [\epsilon^{1-\epsilon^2}R^{-2}\exp(-\epsilon^{3/4})]\mu_2$, so that $\mu_2\bar{\Gamma}_t^{(2)} \leq \mu_3\bar{\Gamma}_t^{(3)}$ when $\psi_t = \epsilon/4$. By (8.72), $\mu_3\bar{\Gamma}_t^{(3)} \leq \mu_2\bar{\Gamma}_t^{(2)}$ when $\psi_t = \epsilon$. $\square$

Proposition 8.13. *Let $\epsilon \in (0, \epsilon_0)$ and ϵ_4 be as in Proposition 8.12, define the following sets:*

$$\begin{aligned}U_0 &= \big\{z \in \mathcal{D} : \psi(z) \leq \epsilon_4\big\}\\ U_1 &= \big\{z \in \mathcal{D} : \epsilon_4 \leq \psi(z) \leq \epsilon/2\big\}\\ U_2 &= \big\{z \in \mathcal{D} : \epsilon/4 \leq \psi(z) \leq \epsilon\big\}\\ U_3 &= \big\{z \in \mathcal{D} : \psi(z) \geq \epsilon\big\}.\end{aligned}$$

Let γ be a smooth path from $[-1,1]$ into U_3, s be some fixed point in $(-1,1)$ and $(Z_t^s)_{t\geq 0}$ be the solution of (8.1) with $\gamma(s)$ as initial condition.

Define as well $(\tau_n)_{n\geq 1}$ *as the sequence of exit times of the process* $(\psi(Z_t^s))_{t\geq 0}$ *from the sets* $[\epsilon/4,+\infty)$, $[\epsilon_4,\epsilon]$ *and* $[0,\epsilon/2]$, *i.e.*

$$\begin{aligned}
&\tau_1 := \inf\{t \geq 0 : \psi_t = \psi(Z_t^s) \leq \epsilon/4\},\\
&\tau_2 := \inf\{t > \tau_1 : \psi_t \notin]\epsilon_4,\epsilon[\},\\
&\tau_3 := \inf\{t > \tau_2 : \psi_t \notin [0,\epsilon/2[\} \quad \text{if } \psi_{\tau_2} = \epsilon_4,\\
&\tau_3 := \inf\{t > \tau_2 : \psi_t \leq \epsilon/4\} \quad \text{if } \psi_{\tau_2} = \epsilon,\\
&\dots
\end{aligned}$$

(If $\tau_n = +\infty$, *then* $\tau_{n+1} = +\infty$ *as well,* $n \geq 1$.*)*

For initial conditions of the form $\gamma(s+\varepsilon)$, *consider the perturbed version* $(\hat{Z}_t^{s+\varepsilon})_{0\leq t\leq \tau_1}$ *as in Proposition 8.10 (*$\epsilon/4$ *playing the role of* ϵ*) up to time* τ_1. *If* $\tau_1 < +\infty$, *extend the perturbed process as* $(\hat{Z}_t^{s+\varepsilon})_{\tau_1\leq t\leq\tau_2}$ *according to the perturbation of Proposition 8.11 (*$\epsilon/2$ *playing the role of* ϵ, ϵ' *being equal to* ϵ_4 *and* λ *to* ϵ^2*) up to time* τ_2. *And so on... according to Figure 8.13 below.*

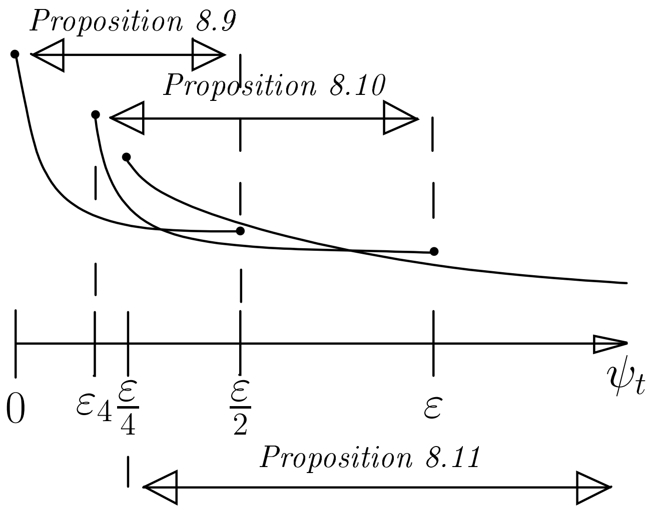

Figure 8.13. Choice of the perturbations.

Assume that the whole process $(\hat{Z}_t^{s+\varepsilon})_{t\geq 0}$ *is differentiable in the mean w.r.t.* ε *and that the derivative process* $(\zeta_t = (d/d\varepsilon)[\hat{Z}_t^{s+\varepsilon}]_{|\varepsilon=0})_{t\geq 0}$ *satisfies the SDE obtained by differentiation of the coefficients of the perturbations as in Theorem 7.2. Then, from time* 0 *to time* τ_1, *consider as* derivative quantity *the process* $(\mu_3\bar{\Gamma}_t^{(3)})_{0\leq t\leq\tau_1}$ *defined in Proposition 8.10. From time* τ_1 *(if finite) to time* τ_2, *consider as* derivative quantity *the process* $(\mu_2\bar{\Gamma}_t^{(2)})_{\tau_1<t\leq\tau_2}$ *defined in Proposition 8.11. And so on... according to Figure 8.8. Denote by* $(\bar{\Gamma}_t)_{t\geq 0}$ *the resulting global* derivative quantity. *(So that the process is left-continuous.)*

Then, we can find $\alpha \in (0,1)$, *depending on* $(\mathbf{A})$ *and* ϵ *only, such that*

$$\mathbb{E}\left[\bar{\Gamma}_t \exp\left(\int_0^t \alpha L\psi(Z_r^s)dr\right)\right] \leq \bar{\Gamma}_0, \quad t \geq 0.$$

Moreover, there exists a constant $C \geq 0$, depending on $(\mathbf{A})$ and ϵ only, such that

$$\mathbb{E}\left[|\zeta_t|^2 \exp\left(\int_0^t \alpha L\psi_r dr\right)\right] \leq C\bar{\Gamma}_0, \quad t \geq 0. \tag{8.73}$$

Proof. By Proposition 8.10, we can find some exponent $\alpha < 1$ (depending on $(\mathbf{A})$ and ϵ only) such that

$$d\left[\bar{\Gamma}_t^{(3)} \exp\left(\int_0^t \alpha L\psi_r dr\right)\right] \leq dm_t, \quad 0 \leq t \leq \tau_1, \tag{8.74}$$

$(m_t)_{t\geq 0}$ standing for a generic martingale term (whose value may vary from line to line).

Consider the case when $\tau_1 < +\infty$. By Proposition 8.11, we can modify α so that

$$d\left[\bar{\Gamma}_t^{(2)} \exp\left(\int_0^t \alpha L\psi_r dr\right)\right] \leq dm_t, \quad \tau_1 \leq t \leq \tau_2 \tag{8.75}$$

We then gather both *derivative quantities* $(\mu_3\bar{\Gamma}_t^{(3)})_{0\leq t\leq \tau_1}$ and $(\mu_2\bar{\Gamma}_t^{(2)})_{\tau_1\leq t\leq \tau_2}$ into a single one, denoted by $(\bar{\Gamma}_t)_{0\leq t\leq \tau_2}$. Obviously, it may be discontinuous at time τ_1: by convention, we assume it to be left-continuous so that $\bar{\Gamma}_{\tau_1} = \mu_3\bar{\Gamma}_{\tau_1}^{(3)}$. Then, we can rewrite (8.74) and (8.75) as

$$\begin{aligned}
&\mathbb{E}\left[\mu_3\bar{\Gamma}_{\tau_1}^{(3)} \exp\left(\int_0^{\tau_1} \alpha L\psi_r dr\right)\mathbf{1}_{\{\tau_1<+\infty\}}\right] \leq \mu_3\bar{\Gamma}_0^{(3)} = \bar{\Gamma}_0 \\
&\mathbb{E}\left[\mu_2\bar{\Gamma}_{t\wedge\tau_2}^{(2)} \exp\left(\int_0^{t\wedge\tau_2} \alpha L\psi_r dr\right)|\mathcal{F}_{\tau_1}\right]\mathbf{1}_{\{\tau_1<+\infty\}} \\
&\qquad \leq \mu_2\bar{\Gamma}_{t\wedge\tau_1}^{(2)} \exp\left(\int_0^{t\wedge\tau_1} \alpha L\psi_r dr\right)\mathbf{1}_{\{\tau_1<+\infty\}}.
\end{aligned} \tag{8.76}$$

(The second inequality above is obviously true if $t \leq \tau_1$: in that case, everything is known at time $t\wedge\tau_2$ and the conditional expectation is useless. Otherwise, i.e. if $t > \tau_1$, the second inequality is a consequence of (8.75). Add also that $\{\tau_1 < +\infty\} \in \mathcal{F}_{\tau_1}$: at time τ_1, τ_1 is known to be finite or not.)

We now apply Proposition 8.12. If $\tau_1 < +\infty$ and $t > \tau_1$, we know that $\psi_{t\wedge\tau_1} = \psi_{\tau_1} = \epsilon/4$ so that $\mu_2\bar{\Gamma}_{\tau_1}^{(2)} \leq \mu_3\bar{\Gamma}_{\tau_1}^{(3)}$. Then, for $t > \tau_1$ (and $\tau_1 < +\infty$), (8.76) yields

$$\mathbb{E}\left[\mu_2\bar{\Gamma}_{t\wedge\tau_2}^{(2)} \exp\left(\int_0^{t\wedge\tau_2} \alpha L\psi_r dr\right)|\mathcal{F}_{\tau_1}\right] \leq \mu_3\bar{\Gamma}_{\tau_1}^{(3)} \exp\left(\int_0^{\tau_1} \alpha L\psi_r dr\right),$$

i.e.

$$\mathbb{E}\left[\bar{\Gamma}_{t\wedge\tau_2} \exp\left(\int_0^{t\wedge\tau_2} \alpha L\psi_r dr\right)|\mathcal{F}_{\tau_1}\right] \leq \bar{\Gamma}_{\tau_1} \exp\left(\int_0^{\tau_1} \alpha L\psi_r dr\right).$$

Finally, for any $t \geq 0$,

$$\begin{aligned}
&\mathbb{E}\Big[\bar{\Gamma}_{t\wedge\tau_2}\exp\Big(\int_0^{t\wedge\tau_2}\alpha L\psi_r dr\Big)\Big] \\
&= \mathbb{E}\Big[\bar{\Gamma}_{t\wedge\tau_2}\exp\Big(\int_0^{t\wedge\tau_2}\alpha L\psi_r dr\Big)\mathbf{1}_{\{\tau_1<t\}}\Big] \\
&\quad + \mathbb{E}\Big[\bar{\Gamma}_{t\wedge\tau_2}\exp\Big(\int_0^{t\wedge\tau_2}\alpha L\psi_r dr\Big)\mathbf{1}_{\{\tau_1\geq t\}}\Big] \\
&= \mathbb{E}\Big[\mathbb{E}\Big[\bar{\Gamma}_{t\wedge\tau_2}\exp\Big(\int_0^{t\wedge\tau_2}\alpha L\psi_r dr\Big)\mathbf{1}_{\{\tau_1<t\}}\Big]\big|\mathcal{F}_{\tau_1}\Big] \\
&\quad + \mathbb{E}\Big[\mathbb{E}\Big[\bar{\Gamma}_{t\wedge\tau_2}\exp\Big(\int_0^{t\wedge\tau_2}\alpha L\psi_r dr\Big)\mathbf{1}_{\{\tau_1\geq t\}}\Big]\big|\mathcal{F}_{\tau_1}\Big].
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\mathbb{E}\Big[\bar{\Gamma}_{t\wedge\tau_2}\exp\Big(\int_0^{t\wedge\tau_2}\alpha L\psi_r dr\Big)\Big] \\
&\leq \mathbb{E}\Big[\bar{\Gamma}_{\tau_1}\exp\Big(\int_0^{\tau_1}\alpha L\psi_r dr\Big)\mathbf{1}_{\{\tau_1<t\}}\Big] \\
&\quad + \mathbb{E}\Big[\bar{\Gamma}_{t\wedge\tau_1}\exp\Big(\int_0^{t\wedge\tau_1}\alpha L\psi_r dr\Big)\mathbf{1}_{\{\tau_1\geq t\}}\Big] \\
&= \mathbb{E}\Big[\bar{\Gamma}_{t\wedge\tau_1}\exp\Big(\int_0^{t\wedge\tau_1}\alpha L\psi_r dr\Big)\Big] \leq \bar{\Gamma}_0.
\end{aligned}$$

In other words, we are able to gather the two inequalities in (8.76) into a single one over the whole interval $[0,\tau_2)$. By induction, we can process further: if $\tau_2 < +\infty$ and $\psi_{\tau_2} = \epsilon_4$, we make use of Proposition 8.9 up to $\tau_3 = \inf\{t > \tau_2 : \psi_t \geq \epsilon/2\}$; if $\tau_2 < +\infty$ and $\psi_{\tau_2} = \epsilon$, we make use of Proposition 8.10 up to $\tau_3 = \inf\{t > \tau_2 : \psi_t \leq \epsilon/4\}$; we then extend $\bar{\Gamma}_t$ to $[0,\tau_3)$ by using Proposition 8.12 (at time τ_2, $\mu_2\bar{\Gamma}^{(2)}_{\tau_2}$ is greater than the two other *derivative quantities*); and so on... We then extend the *derivative quantity* to the whole $[0,+\infty)$ in such a way that

$$\mathbb{E}\Big[\bar{\Gamma}_t\exp\Big(\int_0^t\alpha L\psi_r dr\Big)\Big] \leq \bar{\Gamma}_0.$$

Of course, the value of $\bar{\Gamma}_t$ is given by one of the three original *derivative quantities* $\bar{\Gamma}^{(1)}_t$, $\mu_2\bar{\Gamma}^{(2)}_t$ and $\mu_3\bar{\Gamma}^{(3)}_t$ according to the position of Z^s_t in $\mathcal{D}$. (See Figure 8.8.) What is important is that, in any case, $\bar{\Gamma}_t \geq c|\zeta_t|^2$, for some positive c depending on $(\mathbf{A})$ and ϵ only. Eq. (8.73) follows. □

8.9. **Conclusion.** It now remains to gather all the localized value functions into a single one:

Proposition 8.14. *Keep the assumption and notation of Proposition 8.13. (In particular, s stands below for some fixed real in $(-1,1)$.) Given $S>0$ and ε with $s+\varepsilon \in (-1,1)$, define* *the globally perturbed analog of* V *in Proposition 6.9*

$$
\begin{aligned}
&\hat{V}^{\sigma}_S(s+\varepsilon)\\
&= \mathbb{E}\int_0^{+\infty}\Bigg[\exp\bigg(-\int_0^t 2\mathrm{Re}\big[\langle \bar{G}(Z^s_r,\hat{Z}^{s+\varepsilon}_r - Z^s_r), dB_r\rangle\big]\\
&\qquad\qquad\qquad\qquad - \int_0^t |G|^2(Z^s_r,\hat{Z}^{s+\varepsilon}_r - Z^s_r)dr\bigg)\\
&\qquad\quad \times \exp\bigg(\int_0^t |\tau^{\varepsilon}_r|^2\mathrm{Trace}[\exp(p^{\varepsilon}_r)a_r\exp(-p^{\varepsilon}_r)D^2_{z,\bar{z}}\psi(\hat{Z}^{s+\varepsilon}_r)]dr\bigg)\\
&\qquad\quad \times F\big(\det(a_t),\exp(p^{\varepsilon}_t)a_t\exp(-p^{\varepsilon}_t),\hat{Z}^{s+\varepsilon}_t\big)\phi\big(\frac{\mathfrak{T}^{\varepsilon}_t}{S}\big)\Bigg]|\tau^{\varepsilon}_t|^2dt,
\end{aligned}
\tag{8.77}
$$

where the quantities $(p^{\varepsilon}_t = P(Z^s_t,\hat{Z}^{s+\varepsilon}_t - Z^s_t))_{t\geq 0}$, $(\tau^{\varepsilon}_t = T(Z^s_t,\hat{Z}^{s+\varepsilon}_t - Z^s_t))_{t\geq 0}$ *and* $(G(Z^s_t,\hat{Z}^{s+\varepsilon}_t - Z^s_t))_{t\geq 0}$ *stand for the different possible perturbations used in Proposition 8.13. Precisely,* p^{ε} *is set equal to* 0 *outside the intervals on which the perturbation of Proposition 8.2 applies,* τ^{ε} *is set equal to* 1 *outside the intervals on which the perturbation of Proposition 8.4 applies and* $(G(Z^s_t,\hat{Z}^{s+\varepsilon}_t - Z^s_t))_{t\geq 0}$ *is set equal to* 0 *outside the intervals on which the perturbation of Proposition 8.7 applies. Moreover,* $\dot{\mathfrak{T}}^{\varepsilon}_t = |\tau^{\varepsilon}_t|^2$, $t \geq 0$.

Then, at point s, $\sup_{\sigma}\hat{V}^{\sigma}_S(s) = V_S(\gamma(s))$ *exactly, where* $V_S(\gamma(s))$ *stands for the finite-horizon version of* $V(\gamma(s))$ *in Proposition 6.9, i.e.*

$$
V_S(z) = \sup_{\sigma} V^{\sigma}_S(z), \quad z \in \mathcal{D},
$$

where

$$
\begin{aligned}
V^{\sigma}_S(z) = \mathbb{E}\bigg[\int_0^{+\infty}\exp\bigg(\int_0^t \mathrm{Trace}\big[a_r D_{z,\bar{z}}\psi(Z^{\sigma,z}_r)\big]dr\bigg)&\\
\times F(\det(a_t),a_t,Z^{\sigma,z}_t)\phi\big(\frac{t}{S}\big)dt&\bigg].
\end{aligned}
$$

Moreover, for any control $(\sigma_t)_{t\geq 0}$, $\hat{V}^{\sigma}_S(s+\varepsilon) \leq V_S(\gamma(s+\varepsilon))$.

Sketch of the Proof. The equality $\sup_{\sigma}\hat{V}^{\sigma}_S(s) = V_S(\gamma(s))$ is easily seen.

The proof of the inequality $\sup_{\sigma}\hat{V}^{\sigma}_S(s+\varepsilon) \leq V_S(\gamma(s+\varepsilon))$ is a bit more challenging. We won't perform it in a complete way. We refer the reader to the original articles by Krylov [6, 8]: the argument is explained therein in a very detailed way. However the idea is quite clear and consists in coupling the arguments given in Subsections 8.2, 8.3 and 8.4: modification of the control, of the time speed and of the measure. □

Here is the final step:

Proposition 8.15. *Keep the assumption and notation of Propositions 8.13 and 8.14. Assume in addition that, for any* $S > 0$ *and* $s \in [-1,1]$,

$$
\limsup_{\varepsilon\to 0}\sup_{\sigma}\sup_{|\varepsilon'|<|\varepsilon|}\big[\big|\frac{\partial}{\partial\varepsilon'}\big[\hat{V}^{\sigma}_S(s+\varepsilon')\big]\big|\big] = \big|\frac{\partial}{\partial\varepsilon'}\big[\hat{V}^{\sigma}_S(s+\varepsilon')\big]_{|\varepsilon'=0}\big|.
\tag{8.78}
$$

Assume also that, for every compact interval $I \subset (-1,1)$, for ε small enough, the quantity $\sup_\sigma \sup_{|\varepsilon'|<|\varepsilon|}[|(\partial/\partial\varepsilon')[\hat{V}_S^\sigma(s+\varepsilon')]|]$ *is uniformly bounded w.r.t. $s \in I$. (Pay attention that the definition of the function $\hat{V}_S^\sigma$ depends on s itself.)*

Then, there exists a constant $C > 0$, depending on (**A**) *and the distance from $\gamma([-1,1])$ to $\partial\mathcal{D}$ only, such that, for any $S > 0$, the function $s \in (-1,1) \mapsto V_S(\gamma(s)) + C\int_0^s |\gamma'(r)|dr$ is non-decreasing and the function $s \in (-1,1) \mapsto V_S(\gamma(s)) - C\int_0^s |\gamma'(r)|dr$ is non-increasing.*

Proof. Without loss of generality, we can assume ϵ to be small enough so that $\gamma([-1,1]) \subset U_3$, with U_3 as in Proposition 8.13. Following the proofs of Propositions 8.2, 8.4 and 8.7, we then claim that (C being as in the statement)

$$\left|\frac{d}{d\varepsilon}\left[\hat{V}_S^\sigma(s+\varepsilon)\right]_{|\varepsilon=0}\right| \le C\mathbb{E}\left[\int_0^{+\infty} \exp\left(\int_0^t \mathrm{Trace}[a_r D^2_{z,\bar{z}}\psi(Z_r^s)]dr\right) \times \left[|\zeta_t| + \int_0^t (1+r^{-1/2})|\zeta_r|dr\right]dt\right]. \tag{8.79}$$

Recall that $\mathrm{Trace}[a_r D^2_{z,\bar{z}}\psi(Z_r^s)] \le -N = \epsilon^{-1/4}$. By (8.73), we deduce

$$\begin{aligned}
&\left|\frac{d}{d\varepsilon}\left[\hat{V}_S^\sigma(s+\varepsilon)\right]_{|\varepsilon=0}\right| \\
&\le C\mathbb{E}\left[\int_0^{+\infty} \exp\left(-N(1-\alpha/2)t\right)\left[|\hat{\zeta}_t| \exp\left(\int_0^t (\alpha/2)L\psi_r dr\right)\right.\right. \\
&\qquad \left.\left. + \int_0^t (1+r^{-1/2})|\hat{\zeta}_r| \exp\left(\int_0^r (\alpha/2)L\psi_u du\right)dr\right]dt\right] \\
&= C\int_0^{+\infty} \exp\left(-N(1-\alpha/2)t\right)\left\{\mathbb{E}\left[|\hat{\zeta}_t| \exp\left(\int_0^t (\alpha/2)L\psi_r dr\right)\right]\right. \\
&\qquad \left. + \int_0^t (1+r^{-1/2})\mathbb{E}\left[|\hat{\zeta}_r| \exp\left(\int_0^r (\alpha/2)L\psi_u du\right)\right]dr\right\}dt \\
&\le C\bar{\Gamma}_0^{1/2}\int_0^{+\infty} \exp\left(-N(1-\alpha/2)t\right)(1+t)dt \le C\bar{\Gamma}_0^{1/2},
\end{aligned}$$

the last line following from Cauchy-Schwarz inequality.

Since $\bar{\Gamma}_0 = \bar{\Gamma}_0^{(3)}$, we deduce that

$$\left|\frac{d}{d\varepsilon}\left[\hat{V}_S^\sigma(s+\varepsilon)\right]_{|\varepsilon=0}\right| \le CR|\gamma(s)|^{-1/2}|\gamma'(s)|. \tag{8.80}$$

Unfortunately, the above estimate is a bit weaker than (8.5) and is not sufficient to recover

$$\liminf_{\varepsilon\to0,\varepsilon\neq0} \frac{V_S(\gamma(s+\varepsilon)) - V_S(\gamma(s))}{|\varepsilon|} \ge -CR|\gamma(s)|^{-1/2}|\gamma'(s)|, \tag{8.81}$$

as in (8.6).

To recover (8.6), we take benefit of (8.78). Indeed, by the mean value Theorem, we can generalize (8.80) and write (for a possibly new value of the constant C)

$$
\begin{aligned}
V_S(\gamma(s+\varepsilon)) - V_S(\gamma(s)) &\geq \inf_{\sigma}\big[\hat{V}_S^{\sigma}(s+\varepsilon) - \hat{V}_S^{\sigma}(s)\big] \\
&\geq -C|\varepsilon| \sup_{|\varepsilon'|<|\varepsilon|} \sup_{\sigma}\big[\big|\frac{d}{d\varepsilon'}\big[\hat{V}_S(s+\varepsilon')\big]\big|\big].
\end{aligned}
\tag{8.82}
$$

By (8.78) and (8.80), we deduce (8.81). Modifying the constant C in (8.81) (have in mind that C may depend on ϵ but is independent of S and s), we deduce that

$$
\begin{aligned}
\liminf_{\varepsilon\to 0, \varepsilon>0} \varepsilon^{-1}\Bigg[& V_S(\gamma(s+\varepsilon)) + CR\int_0^{s+\varepsilon} |\gamma'(r)|dr \\
& - V_S(\gamma(s)) - CR\int_0^{s} |\gamma'(r)|dr\Bigg] \geq 0.
\end{aligned}
\tag{8.83}
$$

Actually, (8.82) says a little bit more. Since $\sup_{|\varepsilon'|<|\varepsilon|}\sup_{\sigma}[|[d/d\varepsilon'](\hat{V}_S(s+\varepsilon'))|]$ is bounded in s in compact subsets of $(-1,1)$ (at least for $|\varepsilon|$ small enough), we deduce that the function $V_S\circ\gamma$ is Lipschitz continuous and thus continuous. (Pay attention that the Lipschitz constant may depend on S at this stage of the proof.) Indeed, the LHS in (8.82) being bounded from below uniformly in s, the points s and $s+\varepsilon$ may be exchanged, so that the bound holds from above as well.

We then deduce from (8.83) that the function $s \in (-1,1) \mapsto V_S(\gamma(s)) + C\int_0^s |\gamma'(r)|dr$ is non-decreasing.

Letting S tend to $+\infty$, we deduce that the function $s \in (-1,1) \mapsto V(\gamma(s)) + C\int_0^s |\gamma'(r)|dr$ is non-decreasing. Similarly (i.e. by changing ε into $-\varepsilon$), we can prove that the function $s \in (-1,1) \mapsto V(\gamma(s)) - C\int_0^s |\gamma'(r)|dr$ is non-increasing.

To complete the proof of Meta-Theorem 8.1, it remains to choose γ. For some point z such that $\psi(z) > \epsilon$, we can set $\gamma(s) = z + s\nu$, $s \in [-1,1]$, for some $\nu \in \mathbb{C}^d$ such that the complex closed ball of center z and of radius $|\nu|$ be included in U_3. (See the definition of U_3 in the statement of Proposition 8.13.) Then, $V(\gamma(1)) - V(\gamma(0)) + C|\nu| \geq 0$ and $V(\gamma(1)) - V(\gamma(0)) - C|\nu| \leq 0$, i.e. $|V(z+\nu) - V(z)| \leq C|\nu|$, the constant C here depending on ϵ. Going back to the connection between V and the solution to Monge-Ampère in Proposition 6.9, we understand that the solution to Monge-Ampère is Lipschitz continuous in every compact subset of $\mathcal{D}$. $\square$

Unfortunately, the argument fails for the second-order derivatives. The reason is quite simple. Indeed, we wish to apply Proposition 7.9. Replacing $(\zeta_t)_{t\geq 0}$ by $(\eta_t = [d^2/d\varepsilon^2](\hat{Z}_t^{s+\varepsilon}))_{t\geq 0}$ in the definition of $\bar{\Gamma}_t^{(1)}$, $\bar{\Gamma}_t^{(2)}$ and $\bar{\Gamma}_t^{(3)}$ in (8.71), the problem is to prove that the resulting global second-order *derivative quantity*, denoted by $(\bar{\Gamma}_t(\eta_t))_{t\geq 0}$, satisfies (compare with (8.73))

$$
\mathbb{E}\left[\bar{\Gamma}_t^{1/2}(\eta_t)\exp\left(\int_0^t \alpha L\psi(Z_r^s)dr\right)\right] \leq C\bar{\Gamma}_0^{1/2}, \quad t \geq 0.
$$

In some sense, this matches (7.22) in Proposition 7.9.

The problem is not to prove $\partial\bar{\Gamma}_t(\eta_t) \leq \alpha' L\psi(Z_r^s)\bar{\Gamma}_t(\eta_t)$, $t \geq 0$. (The notation $(\partial\bar{\Gamma}_t(\eta_t))_{t\geq 0}$ has the same meaning as in Proposition 7.9.) Basically, if the inequality is satisfied for ζ_t, it is satisfied for η_t as well: it is sufficient to replace ζ_t by η_t therein. The problem is somewhere else: in Proposition 7.9, the derivative quantity is assumed to be driven by a quadratic form equivalent to the Hermitian (Euclidean in the real case) one. Obviously, this is not the case when using $(\bar{\Gamma}_t(\eta_t))_{t\geq 0}$ since $(\bar{\Gamma}_t^{(1)})_{t\geq 0}$ in (8.71), which is the *derivative quantity* we used in the neighborhood of the boundary, has some singular coefficient inside: $(\psi_t^{-1})_{t\geq 0}$.

9. Proof of the $\mathcal{C}^{1,1}$-Regularity up to the Boundary

We now complete the proof of Theorem 6.1.

In comparison with Section 8, Krylov's program consists in introducing an alternative representation of the solution of the Monge-Ampère equation in the neighborhood of the boundary and to associate a new *derivative quantity* with it, free of any singularities, so that Proposition 7.9 may apply.

9.1. **Representation Process on a Zero Surface.** The trick consists in introducing a parameterized version of Eq. (6.12) in the statement of Proposition 6.9. In what follows, we thus consider the system (with values in $\mathbb{C}^d \times \mathbb{C}^2$)

$$
\begin{aligned}
dZ_t &= \sum_{i=1,2} Y_t^i \sigma_t dB_t^i + a_t D_{\bar{z}}\psi^*(Z_t)dt, \\
dY_t^i &= D_{\bar{z}}\psi(Z_t)\bar{\sigma}_t d\bar{B}_t^i + \frac{1}{2} Y_t^i \mathrm{Trace}\big[a_t D^2_{z,\bar{z}}\psi(Z_t)\big]dt, \quad t\geq 0, \quad i=1,2,
\end{aligned}
\tag{9.1}
$$

where B^1 and B^2 denote two independent complex Brownian motion of dimension d. At that point of the proof, we don't know whether the process $(Z_t)_{t\geq 0}$ stays inside $\mathcal{D}$ or not: since ψ is $\mathcal{C}^4$ in the neighborhood of $\bar{\mathcal{D}}$, we can extend it to the whole $\mathbb{C}^d$ into a $\mathcal{C}^4$ bounded function with bounded derivatives. For such an extension and for a given initial condition (Z_0, Y_0), the above system has locally Lipschitz coefficients and is therefore uniquely solvable on some interval $[0,\tau)$, τ here standing for a stopping time.

In what follows, we set $\Phi(z,y) = \psi(z) - |y|^2$ for $z \in \mathbb{C}^d$ (ψ being extended to the whole space) and $y \in \mathbb{C}^2$. We prove below that, for $Z_0 \in \mathcal{D}$, the solution $(Z_t, Y_t)_{0\leq t<\tau}$ lives in a level set of the function Φ so that it can be extended to the whole $[0,+\infty)$, i.e. $\tau = +\infty$. (Indeed, the level set property says that $(Y_t)_{0\leq t<\tau}$ is bounded by a universal constant.) To do so, we compute for $0 \leq t < \tau$:

$$
\begin{aligned}
d\psi(Z_t) &= \sum_{i=1,2} Y_t^i D_z\psi(Z_t)\sigma_t dB_t^i + \sum_{i=1,2} \bar{Y}_t^i D_{\bar{z}}\psi(Z_t)\bar{\sigma}_t d\bar{B}_t^i \\
&\quad + 2D_z\psi(Z_t) a_t D_{\bar{z}}\psi^*(Z_t)dt + |Y_t|^2 \mathrm{Trace}(a_t D^2_{z,\bar{z}}\psi(Z_t))dt.
\end{aligned}
\tag{9.2}
$$

Above, $|Y_t|^2 = |Y_t^1|^2 + |Y_t^2|^2$. Now, we write for $i \in \{1,2\}$ and $0 \leq t < \tau$:

$$
\begin{aligned}
d|Y_t^i|^2 &= Y_t^i D_z\psi(Z_t)\sigma_t dB_t^i + \bar{Y}_t^i D_{\bar{z}}\psi(Z_t)\bar{\sigma}_t d\bar{B}_t^i \\
&\quad + |Y_t^i|^2 \mathrm{Trace}\big[a_t D^2_{z,\bar{z}}\psi(Z_t)\big]dt + D_z\psi(Z_t) a_t D_{\bar{z}}\psi^*(Z_t)dt.
\end{aligned}
\tag{9.3}
$$

As a consequence, we obtain that

$$
d\big(\psi(Z_t) - |Y_t|^2\big) = 0, \quad 0 \leq t < \tau, \tag{9.4}
$$

so that the process $(\psi(Z_t) - |Y_t|^2)_{0\leq t<\tau}$ lives on a level set of the function Φ. Therefore, $(Y_t)_{0\leq t<\tau}$ is bounded by some universal constant, so that Eq. (9.1) appears as a Lipschitz system.

It now remains to understand how the dynamics of (Z,Y) are connected with the original ones of Z in (6.12). To this end, we set

$$W_t = \sum_{i=1,2} \int_0^t \Big(\frac{Y_s^i}{|Y_s|} \mathbf{1}_{\{|Y_s|>0\}} + \frac{1}{\sqrt{2}} \mathbf{1}_{\{|Y_s|=0\}} \Big) dB_s^i, \quad t \geq 0. \tag{9.5}$$

Clearly, $(W_t)_{t\geq 0}$ is a martingale with values in $\mathbb{C}^d$. Actually, for any coordinates $1 \leq j,k \leq d$,

$$d[W_t^j W_t^k] = 0, \quad d[W_t^j \bar{W}_t^k] = \delta_{j,k} dt, \tag{9.6}$$

where $\delta_{j,k}$ stands for the Kronecker symbol. Following Footnote (12), $(W_t)_{t\geq 0}$ is a complex Brownian motion of dimension d. Moreover, (9.5) implies

$$|Y_t| dW_t = \sum_{i=1,2} Y_t^i dB_t^i, \quad t \geq 0. \tag{9.7}$$

Choose now $Z_0 \in \mathcal{D}$ and $Y_0 \in \mathbb{C}^2$ such that $\psi(Z_0) = |Y_0|^2$. By (9.4), $\psi(Z_t) = |Y_t|^2$ for any $t \geq 0$ so that (9.7) has the form

$$\psi^{1/2}(Z_t) dW_t = \sum_{i=1,2} Y_t^i dB_t^i, \quad t \geq 0.$$

In particular, $(Z_t)_{t\geq 0}$ satisfies

$$dZ_t = \psi^{1/2}(Z_t)\sigma_t dW_t + a_t D_{\bar{z}}\psi^*(Z_t) dt, \quad t \geq 0, \tag{9.8}$$

i.e. (6.12). Clearly, Eq. (9.8) says that Proposition 6.7 applies to $(Z_t)_{t\geq 0}$, that is $(Z_t)_{t\geq 0}$ does not leave $\mathcal{D}$, and that we can use the parameterized version (9.1) of (6.12) in Proposition 6.9. (See Footnote (13) as well.) When doing so, the representation formula holds at some point $z \in \mathcal{D}$: it is the initial condition of Z. However, we stress out that the right initial condition of Eq. (9.1) is the complete initial condition of the pair (Z,Y): given the starting point of Z, the starting point of Y is chosen in such a way that (Z_0, Y_0) is a zero of Φ.

Here is a possible choice:

Proposition 9.1. *Let $\gamma = (\gamma_0, \gamma_1)$ be a smooth path from $[-1,1]$ into $\mathcal{D} \times \mathbb{C}^2$ such that, for any $s \in [-1,1]$, $\Phi(\gamma(s)) = 0$, where $\Phi(z,y) = \psi(z) - |y|^2$, $z \in \mathcal{D}$, $y \in \mathbb{C}^2$. Then, for any $s \in [-1,1]$, the solution $(Z_t^s, Y_t^s)_{t\geq 0}$ to*

$$dZ_t^s = \sum_{i=1,2} (Y_t^s)^i \sigma_t dB_t^i + a_t D_{\bar{z}}\psi^*(Z_t^s) dt,$$
$$d(Y_t^s)^i = D_{\bar{z}}\psi(Z_t^s)\bar{\sigma}_t d\bar{B}_t^i + \frac{1}{2}(Y_t^s)^i \mathrm{Trace}\big[a_t D_{z,\bar{z}}^2 \psi(Z_t^s)\big] dt, \quad t \geq 0, \ i = 1,2,$$

with $(Z_0^s, Y_0^s) = \gamma(s)$ as initial condition, stays in the zero surface of Φ. (Above, $(B_t^1)_{t\geq 0}$ and $(B_t^2)_{t\geq 0}$ stand for two independent complex Brownian motions of dimension d.)

Moreover, the value function V in Proposition 6.9 may be represented at point $\gamma(s)$ as the supremum of $V^\sigma(\gamma(s))$ obtained by plugging the above choice for $(Z_t^s)_{t\geq 0}$ into the definition of Proposition 6.9.

A possible choice for γ is $\gamma_0(s) = z + s\nu$, $z \in \mathcal{D}$ and $\nu \in \mathbb{C}^d \setminus \{0\}$ (such that $B(z,|\nu|) \subset \mathcal{D}$) and $\gamma_1 = (\gamma_{1,1}, \gamma_{1,2})$ solution of the ODE

$$\dot{\gamma}_{1,1}(s) = \bar{\gamma}_{1,1}^{-1}(s) D_z\psi(\gamma_0(s))\nu, \quad \dot{\gamma}_{1,2}(s) = 0, \quad s \in [-1,1], \tag{9.9}$$

with $|\gamma_{1,1}(0)|^2 = \psi(z)$ *and* $\gamma_{1,2}(0) = 0$.

Proof. The first part of the statement has been already proven. Turn now to the ODE (9.9). It is solvable on a short time interval around zero as soon as $\gamma_1(0)$ is non zero. Actually, a simple computation shows that, in the neighborhood of 0,

$$\frac{d\big[|\gamma_{1,1}(s)|^2 - \psi(\gamma_0(s))\big]}{ds} = 2\mathrm{Re}\big[D_z\psi(\gamma_0(s))\nu\big] - 2\mathrm{Re}\big[D_z\psi(\gamma_0(s))\nu\big] = 0,$$

so that $|\gamma_{1,1}(s)|^2 = \psi(\gamma_0(s))$ for s in the neighborhood of 0. As $\psi(\gamma_0(s))$ doesn't vanish for $s \in [-1,1]$, γ_1 may be defined on the whole $[-1,1]$ (at least). □

Below, the objective is to compute the derivatives of the pair $(Z_t^s, Y_t^s)_{t\geq 0}$ and to consider a suitable derivative quantity for it. Specifically, we emphasize that the situation is different from the original one in Proposition 6.9: here, the coefficients of the SDE of the pair $(Z_t^s, Y_t^s)_{t\geq 0}$ are smooth up to the boundary. (Because of the exponent 1/2 in ψ, they are not in the original Proposition 6.9.)

9.2. **Example: Estimate on a Ball.** To explain how things work, we first focus on the specific case when the domain is a ball, say the ball of center 0 and radius R. In such a case, we may choose $\psi(z) = R^2 - |z|^2$ so that Eq. (6.12) has the form

$$dZ_t = \big[R^2 - |Z_t|^2\big]^{1/2}\sigma_t dB_t - a_t Z_t dt, \tag{9.10}$$

with $Z_0 = z \in B(0,R) = \{z' \in \mathbb{C}^d : |z'|^2 < R^2\}$.

We then apply Proposition 9.1 with $\Phi(z,y) = \psi(z) - |y|^2 = R^2 - |z|^2 - |y|^2$, $z \in B(0,R)$ and $y \in \mathbb{C}^2$. The parameterized version (9.1) of (9.10) has the form:

$$\begin{aligned} dZ_t &= \sum_{i=1,2} Y_t^i \sigma_t dB_t^i - a_t Z_t dt \\ dY_t^i &= -\langle Z_t, \bar{\sigma}_t d\bar{B}_t^i\rangle - \frac{1}{2}Y_t^i dt, \quad i = 1,2, \end{aligned} \tag{9.11}$$

where $(B_t^1)_{t\geq 0}$ and $(B_t^2)_{t\geq 0}$ are two independent Brownian motions with values in $\mathbb{C}^d$.

We are now in position to complete the analysis on a ball. To do so, we compute the derivatives of the pair (Z,Y): specifically, we initialize the pair at some $\gamma(s)$, s in the neighborhood of zero and for some curve γ on a level set of Φ. (Choose for example γ as in (9.9).) The resulting pair (Z,Y) is denoted by (Z^s, Y^s) as above. The derivative process is denoted by (ζ_t^s, ϱ_t^s). It is understood as ξ^s with the notations of Theorem 7.2. Eq. (9.11) being linear, Theorem 7.2 applies and we obtain:

$$\begin{aligned} d\zeta_t^s &= \sum_{i=1,2} (\varrho_t^s)^i \sigma_t dB_t^i - a_t \zeta_t dt \\ d(\varrho_t^s)^i &= -\langle \zeta_t, \bar{\sigma}_t d\bar{B}_t^i\rangle - \frac{1}{2}(\varrho_t^s)^i dt, \quad i = 1,2. \end{aligned}$$

Have in mind that $d(|Z_t|^2 + |Y_t|^2) = d(-R^2 + |Z_t|^2 + |Y_t|^2) = d[-\psi(Z_t) + |Y_t|^2] = 0$. Similarly, the pair $(\zeta_t^s, \varrho_t^s)_{t\geq 0}$ satisfies

$$d\big(|\zeta_t^s|^2 + |\varrho_t^s|^2\big) = 0.$$

In comparison with Definition 7.6, this means that the *derivative quantity* is zero, i.e.

$$d\Gamma_t^s = 0, \quad t \geq 0,$$

with $\Gamma_t^s = |\xi_t^s|^2 = |\zeta_t^s|^2 + |\varrho_t^s|^2$. In particular,

$$\exp\Bigl(- \int_0^t c_s ds \Bigr) |\xi_t^s|^2 = \exp(-t) |\xi_0^s|^2,$$

where $c_t = -\mathrm{Trace}[a_t D^2_{z,\bar z} \psi(Z_t)] = 1$.

We then recover the conclusion of Proposition 8.13 but the constant C in (8.73) we now obtain is independent of the distance from γ to the boundary $\partial \mathcal{D}$. Moreover, the matrix A in Proposition 7.9 is simply the identity matrix so that a similar bound is expected for the square-root of the second-order *derivative quantity*. This makes the whole difference with Section 8.

9.3. **Perturbed Version.** Obviously, the case of the ball is very specific. In the general case, we go back to the perturbation strategy developed in Section 8 but for the pair (Z, Y) solution of (9.1).

Specifically, we consider a $\mathcal{C}^2$ curve $\gamma : s \in [-1,1] \mapsto \gamma(s)$ such that $\Phi(\gamma(s)) = 0$ for any $s \in [-1,1]$. For a given (fixed) $s \in (-1,1)$ and for ε in the neighborhood of 0, we denote by $(Z_t^{s+\varepsilon}, Y_t^{s+\varepsilon})_{t \geq 0}$ the solution of[24]

$$\begin{aligned} dZ_t^{s+\varepsilon} &= \sum_{i=1,2} (Y_t^{s+\varepsilon})^i \exp(p_t^\varepsilon) \sigma_t dB_t^i + \exp(p_t^\varepsilon) a_t \exp(-p_t^\varepsilon) D_{\bar z}^* \psi(Z_t^{s+\varepsilon}) dt, \\ d(Y_t^{s+\varepsilon})^i &= D_{\bar z} \psi(Z_t^{s+\varepsilon}) \exp(\bar p_t^\varepsilon) \bar\sigma_t d\bar B_t^i \\ &\quad + \frac{1}{2} (Y_t^{s+\varepsilon})^i \mathrm{Trace}\bigl[\exp(p_t^\varepsilon) a_t \exp(-p_t^\varepsilon) D^2_{z,\bar z} \psi(Z_t^{s+\varepsilon}) \bigr] dt, \\ &\qquad\qquad\qquad\qquad\qquad t \geq 0, \quad i = 1,2, \end{aligned} \tag{9.12}$$

with the initial condition $(Z_0^{s+\varepsilon}, Y_0^{s+\varepsilon}) = \gamma(s+\varepsilon)$

Here, the process $(p_t^\varepsilon)_{t \geq 0}$ denotes a ghost parameter with values into the set of anti-Hermitian matrices, exactly as in Eq (8.10). Specifically, $p_t^{s+\varepsilon} = P(Z_t^s, Z_t^{s+\varepsilon} - Z_t^s)$ as in (8.10) with P as in (8.11). As in Subsection 9.1, ψ is here extended to the whole $\mathbb{C}^d$ into a $\mathcal{C}^4$ function with bounded derivatives, so that the above system has Lipschitz coefficients on the whole space and is therefore uniquely solvable for any given initial condition (Z_0, Y_0).

Following the proof of Proposition 9.1, we can compute $d(\psi(Z_t^{s+\varepsilon}) - |Y_t^{s+\varepsilon}|^2)$ for any $t \geq 0$ and prove that it is zero, so that the process $(\psi(Z_t^{s+\varepsilon}) - |Y_t^{s+\varepsilon}|^2)_{t \geq 0}$ lives on the zero set of the function $\Phi : (z,y) \in \mathcal{D} \times \mathbb{C}^2 \mapsto \psi(z) - |y|^2$. (In particular, $(Z_t^{s+\varepsilon})_{t \geq 0}$ does not leave $\mathcal{D}$.)

Here is the analog of Propositions 8.2 and 8.3

Proposition 9.2. *Let $S > 0$ be a positive real, ϕ be a smooth function from $\mathbb{R}_+$ to $[0,1]$ matching 1 on $[0,1]$ and 0 outside $[0,2]$, $\epsilon > 0$ be a small enough real such that $|D_z \psi(z)| > 0$ for $\psi(z) \leq \epsilon$ and $\mathfrak{s}$ be some (finite) stopping time such that $\psi(Z_{\mathfrak{s}}^s) < \epsilon$. For $\mathfrak{t} := \inf\{t \geq \mathfrak{s} : \psi(Z_t^s) \geq \epsilon\}$, consider some process $(Z_t^{s+\varepsilon}, Y_t^{s+\varepsilon})_{0 \leq t \leq \mathfrak{t}}$ for which $([d/d\varepsilon](Z_t^{s+\varepsilon})_{|\varepsilon=0})_{0 \leq t \leq \mathfrak{t}}$ and $([d^2/d\varepsilon^2](Z_t^{s+\varepsilon})_{|\varepsilon=0})_{0 \leq t \leq \mathfrak{t}}$ exist and for which the perturbed SDE (9.12) holds from $\mathfrak{s}$ to $\mathfrak{t}$ and*

[24]For more simplicity, we forget the symbol "ˆ" used in Subsection 8.2.

define

$$
\begin{aligned}
&\hat{V}_S^{\sigma,\mathfrak{s},\mathfrak{t}}(s+\varepsilon)\\
&= \mathbb{E} \int_{\mathfrak{s}}^{\mathfrak{t}} \biggl[\exp\Bigl(\int_0^t \mathrm{Trace}[\exp(p_r^{s+\varepsilon}) a_r \exp(-p_r^{s+\varepsilon}) D^2_{z,\bar{z}} \psi(Z_r^{s+\varepsilon})] dr \Bigr) \\
&\qquad \times F\bigl(\det(a_t), \exp(p_t^{s+\varepsilon}) a_t \exp(-p_t^{s+\varepsilon}), Z_t^{s+\varepsilon}\bigr) \phi\bigl(\frac{t}{S}\bigr) \biggr] dt,
\end{aligned}
$$

with $p_t^{s+\varepsilon} = P(Z_t^s, Z_t^{s+\varepsilon} - Z_t^s)$, $\mathfrak{s} \leq t \leq \mathfrak{t}$, *$P$ being given by* (8.11).

*Assume that the differentiation operator w.r.t. ε and the expectation and integration symbols can be exchanged in the definition of $\hat{V}_S^{\sigma,\mathfrak{s},\mathfrak{t}}$. Then, we can find a constant $C > 0$, depending on Assumption (**A**) and on ϵ only (in particular, it is independent of C), such that*

$$
\begin{aligned}
&\Bigl| \frac{d}{d\varepsilon} \bigl[\hat{V}_S^{\sigma,\mathfrak{s},\mathfrak{t}}(s+\varepsilon) \bigr] \Bigr| \\
&\quad \leq C \mathbb{E} \biggl[\int_{\mathfrak{s}}^{\mathfrak{t}} \exp\Bigl(\int_0^t \mathrm{Trace}[a_r D^2_{z,\bar{z}}(Z_t^s)] dr \Bigr) \bigl[|\zeta_t^s| + \int_0^t |\zeta_r^s| dr \bigr] dt \biggr],
\end{aligned}
$$

where $\zeta_t^s = [d/d\varepsilon](Z_t^{s+\varepsilon})_{|\varepsilon=0}$.

Similarly,

$$
\begin{aligned}
&\Bigl| \frac{d^2}{d\varepsilon^2} \bigl[\hat{V}_S^{\sigma,\mathfrak{s},\mathfrak{t}}(s+\varepsilon) \bigr] \Bigr| \\
&\leq C \mathbb{E} \biggl[\int_{\mathfrak{s}}^{\mathfrak{t}} \exp\Bigl(\int_0^t \mathrm{Trace}[a_r D^2_{z,\bar{z}}(Z_t^s)] dr \Bigr) \\
&\qquad \times \biggl[|\eta_t^s| + |\zeta_t^s|^2 + \int_0^t |\eta_r^s| dr + \int_0^t |\zeta_r^s|^2 dr + \Bigl(\int_0^t |\zeta_r^s| dr \Bigr)^2 \biggr] dt \biggr],
\end{aligned}
$$

where $\eta_t^s = [d^2/d\varepsilon^2](\hat{Z}_t^{s+\varepsilon})_{|\varepsilon=0}$.

9.4. **Derivative Quantity.** We now prove the analog of Proposition 8.9:

Proposition 9.3. *Keep the assumption and notation of Proposition 9.2. Then, there exists a positive real ϵ_1' such that for $0 < \epsilon < \epsilon_1'$, for $N = K = \epsilon^{-1/4}$, for $\psi = N\psi^0$, where ψ^0 is the* reference plurisuperharmonic function *describing $\mathcal{D}$ such that* $\mathrm{Trace}[aD^2_{z,\bar{z}}\psi^0(z)] \leq -1$, *$z \in \mathcal{D}$, for a stopping time $\mathfrak{s}$ at which $\psi(Z_{\mathfrak{s}}^s) < \epsilon$, the* derivative quantity *obtained by perturbing the control parameter as in* (9.12)

$$
\bar{\Gamma}_t^{(1)} = \exp\bigl(-K\psi(Z_t^s)\bigr) |\zeta_t|^2 + |\rho_t|^2, \quad t \geq \mathfrak{s},
$$

with $\zeta_t = [d/d\varepsilon](Z_t^{s+\varepsilon})_{|\varepsilon=0}$ *and* $\rho_t^s = [d/d\varepsilon](Y_t^{s+\varepsilon})_{|\varepsilon=0}$, *satisfies up to time* $\mathfrak{t} = \inf\{t \geq \mathfrak{s} : \psi(Z_t^s) \geq \epsilon\}$

$$
\begin{aligned}
&\mathbb{E}\biggl[\exp\Bigl(\int_0^{t\wedge\mathfrak{t}} (1-\delta) \mathrm{Trace}[a_r D^2_{z,\bar{z}} \psi(Z_r^s)] dr \Bigr) \bar{\Gamma}^{(1)}_{t\wedge\mathfrak{t}} | \mathcal{F}_{\mathfrak{s}} \biggr] \\
&\quad \leq \exp\Bigl(\int_0^{\mathfrak{s}} (1-\delta) \mathrm{Trace}[a_r D^2_{z,\bar{z}} \psi(Z_r^s)] dr \Bigr) \bar{\Gamma}^{(1)}_{\mathfrak{s}}, \quad t \geq \mathfrak{s},
\end{aligned}
$$

with $\delta = 1/N = \epsilon^{1/4}$.

Proof. The proof is similar to the one of Proposition 8.9. The derivatives of $(Z_t^{s+\varepsilon}, Y_t^{s+\varepsilon})_{t\geq 0}$ with respect to ε at $\varepsilon = 0$ are denoted by

$$\zeta_t = \frac{d}{d\varepsilon}[Z_t^{s+\varepsilon}]_{|\varepsilon=0}, \ \varrho_t = \frac{d}{d\varepsilon}[Y_t^{s+\varepsilon}]_{|\varepsilon=0}, \quad t \geq 0.$$

As $(Y_t^{s+\varepsilon})$ is $\mathbb{C}^2$-valued, so is $(\varrho_t)_{t\geq 0}$. Below, we denote by $(\varrho_t^1)_{t\geq 0}$ and $(\varrho_t^2)_{t\geq 0}$ the two coordinates of $(\varrho_t)_{t\geq 0}$. We also use the following notations:

$$\begin{aligned}
&\psi_t = \psi(Z_t^s), \quad (L\psi)_t = \mathrm{Trace}\big(a_t D^2_{z,\bar{z}}\psi(Z_t^s)\big),\\
&Q_t\zeta_t = \frac{d}{d\varepsilon}\big[P(Z_t^s, Z_t^{s+\varepsilon} - Z_t^s)\big]_{|\varepsilon=0}, \quad t \geq 0.
\end{aligned}$$

Moreover, I_d stands for the identity matrix of size d. By Theorem 7.4, the pair $(\zeta_t, \varrho_t)_{\mathfrak{s}\leq t\leq \mathfrak{t}}$ satisfies the equation[25]:

$$\begin{aligned}
d\zeta_t &= \sum_{i=1,2}\big[\varrho_t^i I_d + Y_t^i Q_t\zeta_t\big]\sigma_t dB_t^i + \big[a_t D_{\bar{z},z}\psi_t\zeta_t + a_t D_{\bar{z},\bar{z}}\psi_t\bar{\zeta}_t\big]dt\\
&+ \big[Q_t\zeta_t a_t D_{\bar{z}}^*\psi_t - a_t Q_t\zeta_t D^*_{\bar{z}}\psi_t\big]dt\\
d\varrho_t^i &= \big[\big(D_{\bar{z},z}\psi_t\zeta_t\big)^* + \big(D_{\bar{z},\bar{z}}\psi_t\bar{\zeta}_t\big)^* - D_{\bar{z}}\psi_t(Q_t\zeta_t)^*\big]\bar{\sigma}_t d\bar{B}_t^i + \frac{1}{2}\varrho_t^i L\psi_t dt\\
&+ \frac{1}{2}Y_t^i\big[D_z(L\psi)_t\zeta_t + D_{\bar{z}}(L\psi)_t\bar{\zeta}_t\big]dt\\
&+ \frac{1}{2}Y_t^i\big[\mathrm{Trace}(Q_t\zeta_t a_t D^2_{z,\bar{z}}\psi_t) - \mathrm{Trace}(a_t Q_t\zeta_t D^2_{z,\bar{z}}\psi_t)\big]dt,\\
&\qquad\qquad\qquad\qquad\qquad \mathfrak{s} \leq t \leq \mathfrak{t}, \quad i = 1,2.
\end{aligned}$$

Using the anti-Hermitian property of $Q_t\zeta_t$, we have:

$$\begin{aligned}
&\overline{\mathrm{Trace}(Q_t\zeta_t a_t D^2_{z,\bar{z}}\psi_t)}\\
&= -\mathrm{Trace}((Q_t\zeta_t)^* a_t^* (D^2_{z,\bar{z}}\psi_t)^*)\\
&= -\mathrm{Trace}(D^2_{z,\bar{z}}\psi_t a_t Q_t\zeta_t) = -\mathrm{Trace}(a_t Q_t\zeta_t D^2_{z,\bar{z}}\psi_t), \quad \mathfrak{s} \leq t \leq \mathfrak{t}.
\end{aligned}$$

Taking the complex conjugate in (8.46), we deduce

$$\begin{aligned}
d\varrho_t^i &= r_t|\zeta_t| D_{\bar{z}}\psi_t\bar{\sigma}_t d\bar{B}_t^i + \frac{1}{2}\varrho_t^i L\psi_t dt\\
&+ Y_t^i \mathrm{Re}\big[D_z(L\psi)_t\zeta_t\big]dt\\
&+ Y_t^i \mathrm{Re}\big[\mathrm{Trace}(Q_t\zeta_t a_t D^2_{z,\bar{z}}\psi_t)\big]dt, \quad \mathfrak{s} \leq t \leq \mathfrak{t}, \quad i = 1,2,
\end{aligned}$$

where $(r_t)_{\mathfrak{s}\leq t\leq \mathfrak{t}}$ stands for a generic process scalar process bounded in terms of $(\mathbf{A})$ only. (The values of $(r_t)_{\mathfrak{s}\leq t\leq \mathfrak{t}}$ may vary from line to line.)

[25]The reader may understand that Theorem 7.4 provides both the form of the equation for the pair $(\zeta_t, \varrho_t)_{\mathfrak{s}\leq t\leq \mathfrak{s}}$ and the differentiability property of the process $(Z_t^{s+\varepsilon}, Y_t^{s+\varepsilon})_{\mathfrak{s}\leq t\leq \mathfrak{t}}$ w.r.t. ε. Indeed, Eq. (9.1) satisfies the assumption of Theorem 7.4: there is no singular term inside contrary to Eq. (8.1). (Since the component Y is bounded, the coefficients may be considered as $\mathcal{C}^2$ coefficients with bounded derivatives.)

We are now in position to compute the norm of the derivative process $((\zeta_t, \varrho_t))_{\mathfrak{s} \le t \le \mathfrak{t}}$.

$$
\begin{aligned}
d|\zeta_t|^2 &= 2\mathrm{Re}\langle \bar\zeta_t, a_t D_{\bar z, z}\psi_t \zeta_t + a_t D_{\bar z, \bar z}\psi_t \bar\zeta_t \rangle dt \\
&\quad + 2\mathrm{Re}\langle \bar\zeta_t, Q_t \zeta_t a_t D^*_{\bar z}\psi_t - a_t Q_t \zeta_t D^*_{\bar z}\psi_t \rangle dt \\
&\quad + \sum_{i=1,2} \mathrm{Trace}\big[\big(\varrho^i I_d + (Y_t^s)^i Q_t \zeta_t \big) a_t \big(\bar\varrho^i I_d - (\bar Y_t^s)^i Q_t \zeta_t \big) \big] dt \\
&\quad + dm_t, \quad \mathfrak{s} \le t \le \mathfrak{t}.
\end{aligned} \tag{9.13}
$$

Similarly,

$$
\begin{aligned}
d|\varrho_t|^2 &= |\varrho_t|^2 L\psi_t dt \\
&\quad + 2\mathrm{Re}\big(\langle \varrho_t, \bar Y_t^s \rangle \big) \big[\mathrm{Re}(D_z (L\psi)_t \zeta_t) + \mathrm{Re}\big(\mathrm{Trace}(Q_t \zeta_t a_t D^2_{z,\bar z}\psi_t) \big) \big] dt \\
&\quad + r_t D_z \psi_t a_t D^*_{\bar z}\psi_t |\zeta_t|^2 dt + dm_t, \quad t \ge 0.
\end{aligned} \tag{9.14}
$$

In what follows, we follow Section 8 and modify the choice of ψ according to the observation we made therein: for any constant $c > 0$, $c\psi$ is again a plurisuperharmonic function describing the domain and we denote by ψ^0 some choice of the plurisuperharmonic function such that, for any Hermitian matrix a of trace 1 and for any $z \in \mathcal{D}$, $\mathrm{Trace}[a D^2_{z,\bar z}\psi^0(z)] \le -1$. Then, we understand ψ as $N\psi^0$ for some free parameter N that will be fixed later on.

As a first application, we can simplify the form of $d|\varrho_t|^2$, or at least we can bound it, for $\mathfrak{s} \le t \le \mathfrak{t}$. To this end, have in mind that $|\psi_t| \le \epsilon$ for $\mathfrak{s} \le t \le \mathfrak{t}$ so that $|D_z \psi_t^0| \ge \kappa$ for some given constant $\kappa > 0$ (for $\mathfrak{s} \le t \le \mathfrak{t}$ and for ϵ small enough). Therefore, from (9.14), we claim

$$
\begin{aligned}
& d|\varrho_t|^2 \\
&\quad = N|\varrho_t|^2 L\psi_t^0 dt + N|\varrho_t||\zeta_t||Y_t^s| r_t dt + N^2 |\zeta_t|^2 \mathcal{E}_t^0 r_t dt + dm_t, \quad \mathfrak{s} \le t \le \mathfrak{t},
\end{aligned} \tag{9.15}
$$

where $(r_t)_{\mathfrak{s} \le t \le \mathfrak{t}}$ is a generic notation for a process, bounded by some constant C depending on $(\mathbf{A})$ and κ only. (The values of $(r_t)_{\mathfrak{s} \le t \le \mathfrak{t}}$ may vary from line to line.) Above, $(\psi_t^0)_{\mathfrak{s} \le t \le \mathfrak{t}}$ is understood as $(\psi^0(Z_t^s))_{\mathfrak{s} \le t \le \mathfrak{t}}$ and $(\mathcal{E}_t^0)_{\mathfrak{s} \le t \le \mathfrak{t}}$ stands for $(\mathcal{E}_t^0 := \langle D^*_z \psi_t^0, a_t D^*_{\bar z}\psi_t^0 \rangle))_{\mathfrak{s} \le t \le \mathfrak{t}}$.

By (8.36),

$$
\begin{aligned}
d|\zeta_t|^2 &= |\varrho_t|^2 dt + |Y_t^s||\varrho_t||\zeta_t| r_t dt + |Y_t^s|^2 |\zeta_t|^2 r_t dt \\
&\quad + N|\zeta_t|^2 \mathcal{E}_t^0 r_t dt + N|\zeta_t|^2 (\mathcal{E}_t^0)^{1/2} r_t dt \\
&\quad + 2\sum_{i=1,2} \mathrm{Re}\big[\langle \bar\zeta_t, \big(\varrho_t^i I_d + (Y_t^s)^i Q_t \zeta_t \big) \sigma_t dB_t^i \rangle \big], \quad \mathfrak{s} \le t \le \mathfrak{t}.
\end{aligned} \tag{9.16}
$$

We now consider the derivative quantity

$$
\bar\Gamma_t = \exp(-K\psi_t)|\zeta_t|^2 + |\varrho_t|^2, \quad \mathfrak{s} \le t \le \mathfrak{t}. \tag{9.17}
$$

for some constant $K > 0$ to be chosen later on.

To compute $(d\bar\Gamma_t)_{\mathfrak{s} \le t \le \mathfrak{t}}$, we first note that

$$
d\psi_t = 2\sum_{i=1,2} (Y_t^s)^i \mathrm{Re}\big[D_z \psi_t \sigma_t dB_t^i \big] + 2\langle D_z \psi_t, a_t D^*_{\bar z}\psi_t \rangle dt + |Y_t^s|^2 L\psi_t dt,
$$

so that

$$
\begin{aligned}
& d\big[\exp(-K\psi_t)\big] \\
& = -2K\exp(-K\psi_t)\sum_{i=1,2}\mathrm{Re}\big[(Y_t^s)^i D_z\psi(Z_t^s)\sigma dB_t^i\big] \\
& \quad + [K^2|Y_t^s|^2 - 2K]\exp(-K\psi_t)\langle D_z\psi_t, a_t D_{\bar z}\psi_t\rangle dt \\
& \quad - K\exp(-K\psi_t)|Y_t^s|^2 L\psi_t dt \\
& = -2K\exp(-K\psi_t)\sum_{i=1,2}\mathrm{Re}\big[(Y_t^s)^i D_z\psi(Z_t^s)\sigma dB_t^i\big] \\
& \quad + N^2[K^2|Y_t^s|^2 - 2K]\exp(-K\psi_t)\mathcal{E}_t^0 dt \\
& \quad - NK\exp(-K\psi_t)|Y_t^s|^2 L\psi_t^0 dt, \quad \mathfrak{s} \le t \le \mathfrak{t}.
\end{aligned}
\tag{9.18}
$$

Therefore, from (9.18) and (9.16),

$$
\begin{aligned}
& d\big[\exp(-K\psi_t)|\zeta_t|^2\big] \\
& = \exp(-K\psi_t)\big[|\varrho_t|^2 + |Y_t^s||\varrho_t||\zeta_t|r_t + |Y_t^s|^2|\zeta_t|^2 r_t \\
& \qquad\qquad + N|\zeta_t|^2\mathcal{E}_t^0 r_t + N|\zeta_t|^2(\mathcal{E}_t^0)^{1/2} r_t\big]dt \\
& \quad + |\zeta_t|^2\exp(-K\psi_t)\big[N^2[K^2|Y_t^s|^2 - 2K]\mathcal{E}_t^0 - NK|Y_t^s|^2 L\psi_t^0\big]dt \\
& \quad + NK\exp(-K\psi_t)\big[|Y_t^s||\zeta_t||\varrho_t|r_t + |Y_t^s|^2|\zeta_t|^2 r_t\big] + dm_t, \quad \mathfrak{s} \le t \le \mathfrak{t}.
\end{aligned}
$$

We are now in position to compute $d\bar\Gamma_t$ for $\mathfrak{s} \le t \le \mathfrak{t}$. To this end, have in mind that $L\psi_t^0 \le -1$ and that $|Y_t^s|^2 = \psi_t \le \epsilon$, $\mathfrak{s} \le t \le \mathfrak{t}$. Then, applying Young's inequality to the term $N(\mathcal{E}_t^0)^{1/2}$, the above equation has the form

$$
\begin{aligned}
& d\big[\exp(-K\psi_t)|\zeta_t|^2\big] \\
& \le \exp(-K\psi_t)\big[|\varrho_t|^2 + C(1+\epsilon^{1/2}+\epsilon)|\xi_t|^2 + C(N+N^2)|\zeta_t|^2\mathcal{E}_t^0\big]dt \\
& \quad + |\zeta_t|^2\exp(-K\psi_t)\big[N^2[K^2\epsilon - 2K]\mathcal{E}_t^0 + CNK\epsilon\big]dt \\
& \quad + NK\exp(-K\psi_t)\big[C\epsilon^{1/2}|\xi_t|^2 + C\epsilon|\xi_t|^2\big] + dm_t, \quad \mathfrak{s} \le t \le \mathfrak{t},
\end{aligned}
\tag{9.19}
$$

where $|\xi_t|^2 = |\zeta_t|^2 + |\varrho_t|^2$. (Actually, $(\xi_t)_{t\ge 0}$ must be understood as the derivative process $(\zeta_t, \varrho_t)_{t\ge 0}$.) Similarly, from (9.15),

$$
d|\varrho_t|^2 \le -N|\varrho_t|^2 dt + CN\epsilon^{1/2}|\xi_t|^2 dt + CN^2|\zeta_t|^2\mathcal{E}_t^0 dt + dm_t \quad \mathfrak{s} \le t \le \mathfrak{t}. \tag{9.20}
$$

Therefore, assuming $\epsilon < 1$ and $N \ge 1$,we deduce from (9.19) and (9.20)

$$
\begin{aligned}
d\bar\Gamma_t & \le \exp(-K\psi_t)(1-N)|\varrho_t|^2 dt \\
& \quad + |\xi_t|^2\big(C' + C'N\epsilon^{1/2} + C'NK\epsilon^{1/2}\big)dt \\
& \quad + |\zeta_t|^2\exp(-K\psi_t)N^2\big[K^2\epsilon - 2K + C'\exp(K\psi_t)\big]\mathcal{E}_t^0 dt \\
& \quad + dm_t, \quad \mathfrak{s} \le t \le \mathfrak{t},
\end{aligned}
$$

the constant C' depending on C only. (In particular, C' is independent of K, N, ϵ, s and t.)

Choose now $K = \epsilon^{-1/4}$. We obtain

$$
\begin{aligned}
d\bar\Gamma_t & \le \exp(-K\psi_t)(1-N)|\varrho_t|^2 dt + 2|\xi_t|^2\big(C' + C'N\epsilon^{1/4}\big)dt \\
& \qquad + |\zeta_t|^2\exp(-K\psi_t)N^2\big[\epsilon^{1/2} - 2\epsilon^{-1/4} + C'\exp(\epsilon^{1/4})\big]\mathcal{E}_t^0 dt + dm_t.
\end{aligned}
$$

Choose ϵ small enough such that $\epsilon^{1/2} - 2\epsilon^{-1/4} + C' \exp(\epsilon^{1/4}) < 0$. Then,

$$d\bar{\Gamma}_t \le \exp(-K\psi_t)(1-N)|\varrho_t|^2 dt + 2|\xi_t|^2 \big(C' + C' N \epsilon^{1/4}\big) dt + dm_t, \quad \mathfrak{s} \le t \le \mathfrak{t}.$$

Finally for $N = \epsilon^{-1/4}$, we obtain:

$$d\bar{\Gamma}_t \le 4C'|\xi_t|^2 dt + dm_t \le 4C' \exp(\epsilon^{1/4}) \bar{\Gamma}_t + dm_t \le 12 C' \bar{\Gamma}_t + dm_t. \tag{9.21}$$

The end of the proof is similar to the one of Proposition 8.9. □

9.5. **Global Derivative Quantity.**

Proposition 9.4. *Let $(B_t^1)_{t\ge 0}$ and $(B_t^2)_{t\ge 0}$ be two independent complex Brownian motions of dimension d, the pair being independent of $(B_t)_{t\ge 0}$. Moreover, let ϵ and ϵ_4 be as in Proposition 8.12, ϵ being less than ϵ_1' in Proposition 9.3 as well, γ_0 be a path from $[-1,1]$ into $\mathcal{D}$ and s be a point in $(-1,1)$ such that $\psi(\gamma_0(s)) > \epsilon$.*

For a given progressively-measurable (w.r.t. the filtration generated by the triple of processes $(B_t, B_t^1, B_t^2)_{t\ge 0}$) control $(\sigma_t)_{t\ge 0}$ with values in the set of complex matrices of size $d\times d$ such that $\mathrm{Trace}(\sigma_t \bar{\sigma}_t^*) = 1$, $t \ge 1$, define $(Z_t^s)_{t\ge 0}$ as follows. Set $\mathfrak{r}_0 = 0$. Up to time $\mathfrak{r}_1 = \{t \ge 0 : \psi_t = \psi(Z_t^s) \le \epsilon_4\}$, define $(Z_t^s)_{0\le t\le \mathfrak{r}_1}$ as the solution of the SDE (8.1) with $\gamma_0(s)$ as initial condition. At time $\mathfrak{r}_1$, set $Y_{\mathfrak{r}_1}^s = (\psi^{1/2}(Z_{\mathfrak{r}_1}^s), 0) \in \mathbb{C}^2$ and then define $(Z_t^s, Y_t^s)_{\mathfrak{r}_1 \le t \le \mathfrak{r}_2}$ (with values into $\mathcal{D}\times\mathbb{C}^2$) up to time $\mathfrak{r}_2 = \{t \ge \mathfrak{r}_1 : \psi_t = \psi(Z_t^s) \ge \epsilon/2\}$ as the solution of (9.1). At time $\mathfrak{r}_2$, define $(Z_t^s)_{\mathfrak{r}_2 \le t \le \mathfrak{r}_3}$ up to time $\mathfrak{r}_3 = \{t \ge \mathfrak{r}_1 : \psi_t = \psi(Z_t^s) \le \epsilon_4\}$ as the solution of the SDE (8.1) and so on..., that is

$$dZ_t^s = \psi^{1/2}(Z_t^s)\sigma_t dB_t + a_t D_{\bar{z}}^* \psi(Z_t^s) dt, \quad t \in [\mathfrak{r}_{2k}, \mathfrak{r}_{2k+1}],\ k \ge 0, \tag{9.22}$$

with $Z_0^s = \gamma(s)$ as initial condition (above, $\mathfrak{r}_0 = 0$), and

$$\begin{aligned} dZ_t^s &= \sum_{i=1,2} (Y_t^s)^i \sigma_t dB_t^i + a_t D_{\bar{z}}^* \psi(Z_t^s) dt \\ d\big(Y_t^s\big)^i &= D_{\bar{z}}\psi(Z_t^s)\bar{\sigma}_t d\bar{B}_t^i \\ &\quad + \frac{1}{2}\big(Y_t^s\big)^i \mathrm{Trace}\big[a_t D_{z,\bar{z}}^2 \psi(Z_t^s)\big] dt, \quad t \in [\mathfrak{r}_{2k+1}, \mathfrak{r}_{2k+2}],\ k \ge 0,\ i = 1,2, \end{aligned} \tag{9.23}$$

with $Y_{\mathfrak{r}_{2k+1}} = (\psi^{1/2}(Z_{\mathfrak{r}_{2k+1}}^s), 0)$ as initial condition.

Define also $(\tau_n)_{n\ge 1}$ as the sequence of exit times of the process $(\psi(Z_t^s))_{t\ge 0}$ from the sets $[\epsilon/4, +\infty)$, $[\epsilon_4, \epsilon]$ and $[0, \epsilon/2]$. When the process $(\psi(Z_t^s))_{t\ge 0}$ belongs to $[\epsilon/4, +\infty)$ consider the perturbation given by Proposition 8.10; when $(\psi(Z_t^s))_{t\ge 0}$ belongs to $[\epsilon_4, \epsilon]$ consider the perturbation given by Proposition 8.11: the perturbation is then given by a process of the form $(Z_t^{s+\varepsilon})_{\mathfrak{r}_{2k} \le t \le \mathfrak{r}_{2k+1}}$, with $k \ge 0$. When $(\psi(Z_t^s))_{t\ge 0}$ belongs to $[0, \epsilon/2]$ consider the perturbation given by Proposition 9.3: the perturbation is then given by a pair of the form $(Z_t^{s+\varepsilon}, Y_t^{s+\varepsilon})_{\mathfrak{r}_{2k+1} \le t \le \mathfrak{r}_{2k+2}}$, $k \ge 0$, with $Y_{\mathfrak{r}_{2k+1}}^{s+\varepsilon} = (\psi^{1/2}(Z_{\mathfrak{r}_{2k+1}}^{s+\varepsilon}), 0)$ as initial condition. Specifically,

$$\begin{aligned} dZ_t^{s+\varepsilon} &= T(Z_t^s, Z_t^{s+\varepsilon} - Z_t^s)\psi^{1/2}(Z_t^{s+\varepsilon}) \exp\big(P(Z_t^s, Z_t^{s+\varepsilon} - Z_t^s)\big) \\ &\quad \times \sigma_t\big(dB_t + G(Z_t^s, Z_t^{s+\varepsilon} - Z_t^s) dt\big) \\ &\quad + |T|^2(Z_t^s, Z_t^{s+\varepsilon} - Z_t^s) \exp\big(P(Z_t^s, Z_t^{s+\varepsilon} - Z_t^s)\big) \\ &\quad\quad \times a_t \exp\big(-P(Z_t^s, Z_t^{s+\varepsilon} - Z_t^s)\big) D_{\bar{z}}^* \psi(Z_t^{s+\varepsilon}) dt, \quad \mathfrak{r}_{2k} \le t \le \mathfrak{r}_{2k+1}, \end{aligned}$$

with $Z_0^{s+\varepsilon} = \gamma(s+\varepsilon)$ *as initial condition, and*

$$
\begin{aligned}
dZ_t^{s+\varepsilon} &= \sum_{i=1}^{2} (Y_t^{s+\varepsilon})^i dB_t^i \\
&\quad + \exp\bigl(P(Z_t^s, Z_t^{s+\varepsilon} - Z_t^s)\bigr) a_t \exp\bigl(-P(Z_t^s, Z_t^{s+\varepsilon} - Z_t^s)\bigr) D_{\bar{z}}^* \psi(Z_t^{s+\varepsilon}) dt \\
d\bigl(Y_t^{s+\varepsilon}\bigr)^i &= D_{\bar{z}}\psi(Z_t^{s+\varepsilon}) \exp\bigl(\bar{P}(Z_t^s, Z_t^{s+\varepsilon} - Z_t^s)\bigr) \bar{\sigma}_t d\bar{B}_t^i \\
&\quad + \frac{1}{2} \bigl(Y_t^{s+\varepsilon}\bigr)^i \mathrm{Trace}\Bigl[\exp\bigl(P(Z_t^s, Z_t^{s+\varepsilon} - Z_t^s)\bigr) \\
&\qquad \times a_t \exp\bigl(-P(Z_t^s, Z_t^{s+\varepsilon} - Z_t^s)\bigr) D_{z,\bar{z}}^2 \psi(Z_t^{s+\varepsilon})\Bigr] dt, \\
&\qquad\qquad\qquad\qquad\qquad\qquad \mathfrak{r}_{2k+1} \leq t \leq \mathfrak{r}_{2k+2}, \quad i = 1, 2,
\end{aligned}
$$

with $Y_{\mathfrak{r}_{2k+1}}^{s+\varepsilon} = (\psi^{1/2}(Z_{\mathfrak{r}_{2k+1}}^{s+\varepsilon}), 0)$ *as initial condition.*

Above, $(P(Z_t^s, Z_t^{s+\varepsilon} - Z_t^s))_{t \geq 0}$*,* $(T(Z_t^s, Z_t^{s+\varepsilon} - Z_t^s))_{t \geq 0}$*, and* $(G(Z_t^s, Z_t^{s+\varepsilon} - Z_t^s))_{t \geq 0}$*, stand for the different possible perturbations used in Propositions 8.10, 8.11 and 9.3. Precisely,* $(P(Z_t^s, Z_t^{s+\varepsilon} - Z_t^s))_{t \geq 0}$ *is set equal to* 0 *outside the intervals on which the perturbation of Proposition 8.2 applies,* $(T(Z_t^s, Z_t^{s+\varepsilon} - Z_t^s))_{t \geq 0}$ *is set equal to* 1 *outside the intervals on which the perturbation of Proposition 8.4 applies and* $(G(Z_t^s, Z_t^{s+\varepsilon} - Z_t^s))_{t \geq 0}$ *is set equal to* 0 *outside the intervals on which the perturbation of Propostion 8.7 applies. As a summary, Picture 9.4 below is the analog of Picture 8.13*

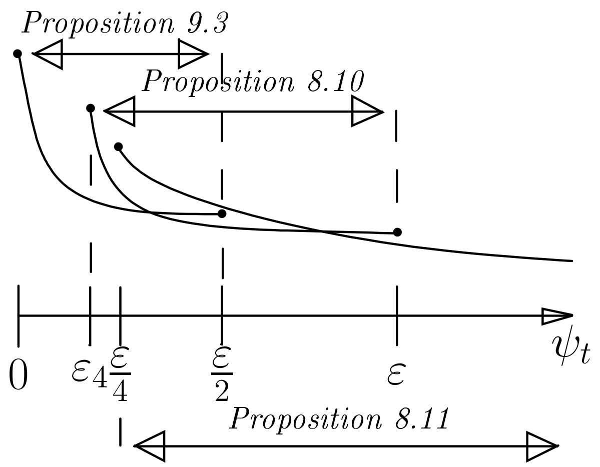

Figure 9.4. Choice of the perturbations with the new representation.

Then, the family of processes $(Z_t^{s+\varepsilon})_{t \geq 0}$*,* ε *in the neighborhood of* 0*, is twice differentiable in probability w.r.t.* ε *at* $\varepsilon = 0$*, with time continuous derivatives. Similarly, for each* $k \geq 0$*, the family of processes* $(Y_t^{s+\varepsilon})_{\mathfrak{r}_{2k+1} \leq t \leq \mathfrak{r}_{2k+2}}$*,* ε *in the neighborhood of* 0*, is twice differentiable in probability w.r.t.* ε *at* $\varepsilon = 0$*, with continuous derivatives. Moreover, the dynamics of the derivatives are obtained by differentiating (w.r.t.* ε*) the dynamics of* $(Z_t^{s+\varepsilon})_{t \geq 0}$ *and* $((Y_t^{s+\varepsilon})_{\mathfrak{r}_{2k+1} \leq t \leq \mathfrak{r}_{2k+2}})_{k \geq 0}$ *formally at* $\varepsilon = 0$*, as done in the* meta-part *of Section 8.*

Define then the derivative quantity $(\bar{\Gamma}_t)_{t \geq 0}$ *as* $\mu_2 \bar{\Gamma}_t^{(2)}$*,* $\mu_3 \bar{\Gamma}_t^{(3)}$ *in Proposition 8.12 and* $\bar{\Gamma}_t^{(1)}$ *in Proposition 9.3. (In particular,* $(\bar{\Gamma}_t)_{t \geq 0}$ *is left-continuous.) Then, we can find* $\alpha \in (0,1)$*,* *depending on* ($\mathbf{A}$) *and* ϵ *only, such that*

$$\mathbb{E}\Big[\bar{\Gamma}_t \exp\Big(\int_0^t \alpha L\psi_r dr\Big)\Big] \le \bar{\Gamma}_0, \quad t \ge 0.$$

Proof. Differentiability properties will be established below. (See Proposition 9.6 below.) In comparison with Subsection 8.8, the only difference is here to show that

$$\lim_{t\to\mathfrak{r}_{2k+1}+} \bar{\Gamma}_t \le \bar{\Gamma}_{\mathfrak{r}_{2k+1}}, \ \lim_{t\to\mathfrak{r}_{2k}+} \bar{\Gamma}_t \le \bar{\Gamma}_{\mathfrak{r}_{2k}} \quad k \ge 0.$$

When $t \to \mathfrak{r}_{2k}+$, $\bar{\Gamma}_t$ is given by $\mu_2\bar{\Gamma}_t^{(2)}$, so that, by Proposition 8.12 (recall that $\psi_{\mathfrak{r}_{2k}} = \epsilon/2$),

$$\begin{aligned} &\lim_{t\to\mathfrak{r}_{2k}+} \bar{\Gamma}_t = \mu_2\bar{\Gamma}^{(2)}_{\mathfrak{r}_{2k}} \\ &= \mu_2 \exp(-\epsilon^{-1/4}\psi_{\mathfrak{r}_{2k}})\psi_{\mathfrak{r}_{2k}}^{-\epsilon^2}|\zeta_{\mathfrak{r}_{2k}}|^2 + 2\mu_2\epsilon^{9/4}\psi_{\mathfrak{r}_{2k}}^{-(1+\epsilon^2)}\mathrm{Re}^2\big[D_z\psi_{\mathfrak{r}_{2k}}\zeta_{\mathfrak{r}_{2k}}\big] \\ &\le \exp(-\epsilon^{-1/4}\psi_{\mathfrak{r}_{2k}})|\zeta_{\mathfrak{r}_{2k}}|^2 + \psi_{\mathfrak{r}_{2k}}^{-1}\mathrm{Re}^2\big[D_z\psi_{\mathfrak{r}_{2k}}\zeta_{\mathfrak{r}_{2k}}\big]. \end{aligned} \tag{9.24}$$

Now, have in mind that $|Y^{s+\varepsilon}_{\mathfrak{r}_{2k}}|^2 = \psi(Z^{s+\varepsilon}_{\mathfrak{r}_{2k}})$ so that, by differentiation,

$$\mathrm{Re}\big[D_z\psi_{\mathfrak{r}_{2k}}\zeta_{\mathfrak{r}_{2k}}\big] = \mathrm{Re}\big[Y^1_{\mathfrak{r}_{2k}}(\bar{\varrho}_{\mathfrak{r}_{2k}})^1\big] + \mathrm{Re}\big[Y^2_{\mathfrak{r}_{2k}}(\bar{\varrho}_{\mathfrak{r}_{2k}})^2\big]. \tag{9.25}$$

Therefore,

$$\big|\mathrm{Re}\big[D_z\psi_{\mathfrak{r}_{2k}}\zeta_{\mathfrak{r}_{2k}}\big]\big| \le |Y^1_{\mathfrak{r}_{2k}}||\varrho^1_{\mathfrak{r}_{2k}}| + |Y^2_{\mathfrak{r}_{2k}}||\varrho^2_{\mathfrak{r}_{2k}}| \le |Y_{\mathfrak{r}_{2k}}||\rho_{\mathfrak{r}_{2k}}|. \tag{9.26}$$

Since $|Y_{\mathfrak{r}_{2k}}| = \psi^{1/2}_{\mathfrak{r}_{2k}}$,

$$\psi^{-1}_{\mathfrak{r}_{2k}}\mathrm{Re}^2\big[D_z\psi_{\mathfrak{r}_{2k}}\zeta_{\mathfrak{r}_{2k}}\big]\big| \le |\varrho_{\mathfrak{r}_{2k}}|^2.$$

From (9.24), we deduce

$$\lim_{t\to\mathfrak{r}_{2k}+} \bar{\Gamma}_t \le \exp(-\epsilon^{-1/4}\psi_{\mathfrak{r}_{2k}})|\zeta_{\mathfrak{r}_{2k}}|^2 + \psi^{-1}_{\mathfrak{r}_{2k}}|\varrho_{\mathfrak{r}_{2k}}|^2 = \bar{\Gamma}_{\mathfrak{r}_{2k}}.$$

It now remains to prove the bound at time $\mathfrak{r}_{2k+1}$. When $t \to \mathfrak{r}_{2k+1}+$, $\bar{\Gamma}_t$ is given by $\bar{\Gamma}^{(1)}_t$, i.e.

$$\bar{\Gamma}_t = \exp(-\epsilon^{-1/4}\psi_t)|\zeta_t|^2 + |\varrho_t|^2.$$

Therefore,

$$\lim_{t\to\mathfrak{r}_{2k+1}+} \bar{\Gamma}_t = \exp(-\epsilon^{-1/4}\psi_{\mathfrak{r}_{2k+1}})|\zeta_{\mathfrak{r}_{2k+1}}|^2 + |\varrho_{\mathfrak{r}_{2k+1}}|^2. \tag{9.27}$$

Have in mind that, at time $t = \mathfrak{r}_{2k+1}$, $Y^{s+\varepsilon}_{\mathfrak{r}_{2k+1}} = (\psi^{1/2}(Z^{s+\varepsilon}_{\mathfrak{r}_{2k+1}}), 0)$, so that, by differentiation,

$$\varrho_{\mathfrak{r}_{2k+1}} = \big(\psi^{-1/2}_{\mathfrak{r}_{2k+1}}\mathrm{Re}\big[D_z\psi_{\mathfrak{r}_{2k+1}}\zeta_{\mathfrak{r}_{2k+1}}\big], 0\big). \tag{9.28}$$

We deduce that

$$\lim_{t\to\mathfrak{r}_{2k+1}+} \bar{\Gamma}_t = \exp(-\epsilon^{-1/4}\psi_{\mathfrak{r}_{2k+1}})|\zeta_{\mathfrak{r}_{2k+1}}|^2 + \psi^{-1}_{\mathfrak{r}_{2k+1}}\big|\mathrm{Re}\big[D_z\psi_{\mathfrak{r}_{2k+1}}\zeta_{\mathfrak{r}_{2k+1}}\big]\big|^2. \tag{9.29}$$

Applying Proposition 8.12 (recall that $\psi_{\mathfrak{r}_{2k+1}} = \epsilon_4$), we obtain

$$\lim_{t\to\mathfrak{r}_{2k+1}+} \bar{\Gamma}_t \le \mu_2\bar{\Gamma}^{(2)}_{\mathfrak{r}_{2k+1}} = \bar{\Gamma}_{\mathfrak{r}_{2k+1}}.$$

This completes the proof. $\square$

We deduce

Corollary 9.5. *Keep the notation of Proposition 9.4 and define the second-order derivatives of $(Z_t^{s+\varepsilon})_{\mathfrak{r}_{2k}\leq t\leq \mathfrak{r}_{2k+1}}$, $k\geq 0$, by setting $\eta_t^s = [d^2/d\varepsilon^2][Z_t^{s+\varepsilon}]_{|\varepsilon=0}$, for $\mathfrak{r}_{2k}\leq t\leq \mathfrak{r}_{2k+1}$, $k\geq 0$, and define the second-order derivatives of $(Z_t^{s+\varepsilon}, Y_t^{s+\varepsilon})_{\mathfrak{r}_{2k+1}\leq t\leq \mathfrak{r}_{2k+2}}$, $k\geq 0$, by setting $(\eta_t^s,\pi_t^s) = [d^2/d\varepsilon^2][(Z_t^{s+\varepsilon}, Y_t^{s+\varepsilon})]_{|\varepsilon=0}$, for $\mathfrak{r}_{2k}\leq t\leq \mathfrak{r}_{2k+1}$, $k\geq 1$.*

Define the analogs of $\bar{\Gamma}_t^{(1)}$, $\mu_2\bar{\Gamma}_t^{(2)}$ and $\mu_3\bar{\Gamma}_t^{(3)}$, $t\geq 0$, i.e.

$$\begin{aligned}
\bar{\Delta}_t^{(1)} &= \exp\bigl(-\epsilon^{-1/4}\psi(Z_t^s)\bigr)|\eta_t|^2 + |\pi_t|^2,\\
\bar{\Delta}_t^{(2)} &= \exp(-\epsilon^{-1/4}\psi_t)\psi_t^{-\epsilon^2}|\eta_t|^2 + 2\epsilon^{9/4}\psi_t^{-(1+\epsilon^2)}\mathrm{Re}^2\bigl[D_z\psi_t\eta_t\bigr],\\
\bar{\Delta}_t^{(3)} &= (R^2-|Z_t|^2)\psi_t^{-1}|\eta_t|^2,
\end{aligned}$$

for some ϵ as in the statement of Proposition 8.12. Define the global second-order derivative quantity $(\bar{\Delta}_t)_{t\geq 0}$ as the analog of $(\bar{\Gamma}_t)_{t\geq 0}$. (In particular, mention that $(\bar{\Delta}_t)_{t\geq 0}$ is left-continuous.)

Then, we can find $\alpha\in(0,1)$ and $C>0$, depending on ($\mathbf{A}$) and ϵ only, such that

$$\mathbb{E}\biggl[\bigl(\bar{\Delta}_t^{1/2}+\bar{\Gamma}_t\bigr)\exp\biggl(\int_0^t \alpha L\psi_r dr\biggr)\biggr] \leq \bar{\Delta}_0^{1/2} + C\bar{\Gamma}_0, \quad t\geq 0.$$

Proof. Following the proof of Proposition 7.9, we can prove that on each $[\tau_n,\tau_{n+1})$, $n\geq 0$, with $\tau_0=0$ and $(\tau_n)_{n\geq 1}$ as in Proposition 9.4, and for any $a>0$,

$$d\biggl[\exp\biggl(\int_0^t \alpha L\psi_r dr\biggr)\bigl(a+\bar{\Delta}_t+\bar{\Gamma}_t^2\bigr)^{1/2}\biggr] \leq C\bar{\Gamma}_t\exp\biggl(\int_0^t \alpha L\psi_r dr\biggr)dt. \tag{9.30}$$

The proof of (9.30) relies on two points. First, what is called $(\partial\bar{\Gamma}_t(X_t^s,(\eta_t^s,\pi_t^s)))_{t\geq 0}$ in the statement of Proposition 7.9 (or equivalently $(\partial\bar{\Delta}_t)_{t\geq 0}$ with the current notation) satisfies the same bound as $(\partial\bar{\Gamma}_t)_{t\geq 0}$. Precisely, $(\partial\bar{\Gamma}_t)_{t\geq 0}$ corresponds to the dt term obtained by differentiating the form $(\bar{\Gamma}_t)_{t\geq 0}$ and then by replacing $(\zeta_t^s,\varrho_t^s)_{t\geq 0}$ therein by $(\eta_t^s,\pi_t^s)_{t\geq 0}$. In the current case, we know that $\partial\bar{\Gamma}_t\leq \alpha L\psi_t\bar{\Gamma}_t$ for any $t\in(\tau_n,\tau_{n+1})$ and for any possible values of the pair $(\zeta_t^s,\varrho_t^s)_{\tau_n\leq t\leq\tau_{n+1}}$. Replacing $(\zeta_t^s,\varrho_t^s)_{\tau_n\leq t\leq\tau_{n+1}}$ by $(\eta_t^s,\pi_t^s)_{\tau_n\leq t\leq\tau_{n+1}}$, we deduce that $\partial\bar{\Delta}_t\leq\alpha L\psi_t\bar{\Delta}_t$ for any $t\in(\tau_n,\tau_{n+1})$. Second, the proof of (9.30) relies on the equivalence of the quadratic form driving $(\bar{\Gamma}_t)_{t\geq 0}$ and $(\bar{\Delta}_t)_{t\geq 0}$ and the current Hermitian form: of (complex) dimension d for $t\in(\mathfrak{r}_{2k},\mathfrak{r}_{2k+1}]$, $k\geq 0$, and of (complex) dimension $d+2$ for $t\in(\mathfrak{r}_{2k+1},\mathfrak{r}_{2k+2}]$. This equivalence makes the difference between Sections 8 and 9.

As a consequence of (9.30), we only need to check the boundary conditions to recover the statement, i.e. we only need to prove that $\lim_{t\to\tau_n+}\bar{\Delta}_t\leq\bar{\Delta}_{\tau_n}$.

If τ_n is different from some $\mathfrak{r}_k$, the result follows from Proposition 8.12.

If τ_n is equal to some $\mathfrak{r}_{2k}$, we follow (9.24). (Keep in mind that $\bar{\Delta}_t$ is given by $\bar{\Delta}_t^{(2)}$ as $t\to\mathfrak{r}_{2k}+$ and by $\bar{\Delta}_t^{(1)}$ as $t\to\mathfrak{r}_{2k}-$.) The point is to bound $\psi_{\mathfrak{r}_{2k}}^{-1}\mathrm{Re}^2[D_z\psi_{\mathfrak{r}_{2k}}\eta_{\mathfrak{r}_{2k}}]$ in terms of $|\pi_{\mathfrak{r}_{2k}}|^2$. We have the analog of (9.25), but with quadratic first-order terms in addition, i.e.

$$\mathrm{Re}\bigl[Y_{\mathfrak{r}_{2k}}^1(\bar{\pi}_{\mathfrak{r}_{2k}})^1\bigr] + \mathrm{Re}\bigl[Y_{\mathfrak{r}_{2k}}^2(\bar{\pi}_{\mathfrak{r}_{2k}})^2\bigr] = \mathrm{Re}\bigl[D_z\psi_{\mathfrak{r}_{2k}}\eta_{\mathfrak{r}_{2k}}\bigr] + O\bigl(|\zeta_{\mathfrak{r}_{2k}}|^2+|\varrho_{\mathfrak{r}_{2k}}|^2\bigr). \tag{9.31}$$

(Here, the constants in the Landau notation $O(\dots)$ only depend on ($\mathbf{A}$).) As in (9.26), we deduce that

$$\begin{aligned}
\psi_{\mathfrak{r}_{2k}}^{-1}\mathrm{Re}^2\bigl[D_z\psi_{\mathfrak{r}_{2k}}\eta_{\mathfrak{r}_{2k}}\bigr] \leq |\pi_{\mathfrak{r}_{2k}}|^2 &+ O\bigl(\bigl|\mathrm{Re}\bigl[D_z\psi_{\mathfrak{r}_{2k}}\eta_{\mathfrak{r}_{2k}}\bigr]\bigr|\bigl(|\zeta_{\mathfrak{r}_{2k}}|^2+|\varrho_{\mathfrak{r}_{2k}}|^2\bigr)\bigr)\\
&+ O\bigl(|\zeta_{\mathfrak{r}_{2k}}|^4+|\varrho_{\mathfrak{r}_{2k}}|^4\bigr).
\end{aligned} \tag{9.32}$$

(Here, the Landau term $O(\dots)$ may depend on ϵ as well. Indeed, $\psi_{\mathfrak{r}_{2k}} = \epsilon/2$.) As a consequence, for any small $a>0$, we can write

$$\begin{aligned}&\psi_{\mathfrak{r}_{2k}}^{-1}\mathrm{Re}^2\big[D_z\psi_{\mathfrak{r}_{2k}}\eta_{\mathfrak{r}_{2k}}\big]\\&\quad\le |\pi_{\mathfrak{r}_{2k}}|^2 + a\psi_{\mathfrak{r}_{2k}}^{-1}\mathrm{Re}^2\big[D_z\psi_{\mathfrak{r}_{2k}}\eta_{\mathfrak{r}_{2k}}\big] + (1+a^{-1})O\big(|\zeta_{\mathfrak{r}_{2k}}|^4 + |\varrho_{\mathfrak{r}_{2k}}|^4\big).\end{aligned}$$

By Proposition 8.12, we then deduce that (recall that $\bar{\Delta}_t$ is given by $\bar{\Delta}_t^{(2)}$ as $t\to\mathfrak{r}_{2k}+$)

$$\begin{aligned}&\lim_{t\to\mathfrak{r}_{2k}+}\bar{\Delta}_t\\&=\mu_2\exp(-\epsilon^{-1/4}\psi_{\mathfrak{r}_{2k}})\psi_{\mathfrak{r}_{2k}}^{-\epsilon^2}|\eta_{\mathfrak{r}_{2k}}|^2 + 2\mu_2\epsilon^{9/4}\psi_{\mathfrak{r}_{2k}}^{-(1+\epsilon^2)}\mathrm{Re}^2\big[D_z\psi_{\mathfrak{r}_{2k}}\eta_{\mathfrak{r}_{2k}}\big]\\&\le \exp(-\epsilon^{-1/4}\psi_{\mathfrak{r}_{2k}})|\eta_{\mathfrak{r}_{2k}}|^2 + \psi_{\mathfrak{r}_{2k}}^{-1}\mathrm{Re}^2\big[D_z\psi_{\mathfrak{r}_{2k}}\eta_{\mathfrak{r}_{2k}}\big]\\&\quad-\big(1-2\epsilon^{9/4}\big)\psi_{\mathfrak{r}_{2k}}^{-1}\mathrm{Re}^2\big[D_z\psi_{\mathfrak{r}_{2k}}\eta_{\mathfrak{r}_{2k}}\big]\\&\le \exp(-\epsilon^{-1/4}\psi_{\mathfrak{r}_{2k}})|\eta_{\mathfrak{r}_{2k}}|^2 + |\pi_{\mathfrak{r}_{2k}}|^2 + \big(a-1+2\epsilon^{9/4}\big)\psi_{\mathfrak{r}_{2k}}^{-1}\mathrm{Re}^2\big[D_z\psi_{\mathfrak{r}_{2k}}\eta_{\mathfrak{r}_{2k}}\big]\\&\quad+(1+a^{-1})O\big(|\zeta_{\mathfrak{r}_{2k}}|^4 + |\varrho_{\mathfrak{r}_{2k}}|^4\big).\end{aligned}$$

Choosing a small enough (in terms of ϵ), we deduce that

$$\lim_{t\to\mathfrak{r}_{2k}+}\bar{\Delta}_t \le \bar{\Delta}_{\mathfrak{r}_{2k}} + C\big(|\zeta_{\mathfrak{r}_{2k}}|^4 + |\varrho_{\mathfrak{r}_{2k}}|^4\big). \tag{9.33}$$

We apply the same strategy when $t\to\mathfrak{r}_{2k+1}+$. (Keep in mind that $\bar{\Delta}_t$ is given by $\bar{\Delta}_t^{(1)}$ as $t\to\mathfrak{r}_{2k+1}+$ and by $\bar{\Delta}_t^{(2)}$ as $t\to\mathfrak{r}_{2k+1}-$.) Following (9.27), we claim

$$\lim_{t\to\mathfrak{r}_{2k+1}+}\bar{\Delta}_t = \exp(-\epsilon^{-1/4}\psi_{\mathfrak{r}_{2k+1}})|\eta_{\mathfrak{r}_{2k+1}}|^2 + |\pi_{\mathfrak{r}_{2k+1}}|^2.$$

Now, as in (9.28),

$$|\pi_{\mathfrak{r}_{2k+1}}| = \psi_{\mathfrak{r}_{2k+1}}^{-1/2}\big|\mathrm{Re}\big[D_z\psi_{\mathfrak{r}_{2k+1}}\eta_{\mathfrak{r}_{2k+1}}\big]\big| + O(|\zeta_{\mathfrak{r}_{2k+1}}|^2).$$

(Here as well, $O(\cdots)$ may depend on ϵ and ϵ_4. Indeed, $\psi_{\mathfrak{r}_{2k+1}} = \epsilon_4$.)

In particular, for any small $a>0$,

$$|\pi_{\mathfrak{r}_{2k+1}}|^2 \le (1+a)\psi_{\mathfrak{r}_{2k+1}}^{-1}\mathrm{Re}^2\big[D_z\psi_{\mathfrak{r}_{2k+1}}\eta_{\mathfrak{r}_{2k+1}}\big] + (1+a^{-1})O(|\zeta_{\mathfrak{r}_{2k+1}}|^4).$$

Following (9.29) and using Proposition 8.12, we deduce (as $t\to\mathfrak{r}_{2k+1}+$, $\bar{\Delta}_t$ is given by $\bar{\Delta}_t^{(1)}$)

$$\begin{aligned}&\lim_{t\to\mathfrak{r}_{2k+1}+}\bar{\Delta}_t\\&\le \exp(-\epsilon^{-1/4}\psi_{\mathfrak{r}_{2k+1}})|\eta_{\mathfrak{r}_{2k+1}}|^2 + \psi_{\mathfrak{r}_{2k+1}}^{-1}\mathrm{Re}^2\big[D_z\psi_{\mathfrak{r}_{2k+1}}\eta_{\mathfrak{r}_{2k+1}}\big]\\&\quad+a\psi_{\mathfrak{r}_{2k+1}}^{-1}\mathrm{Re}^2\big[D_z\psi_{\mathfrak{r}_{2k+1}}\eta_{\mathfrak{r}_{2k+1}}\big] + (1+a^{-1})O\big(|\zeta_{\mathfrak{r}_{2k+1}}|^4\big).\\&\le \mu_2\exp(-\epsilon^{-1/4}\psi_{\mathfrak{r}_{2k}})\psi_{\mathfrak{r}_{2k}}^{-\epsilon^2}|\eta_{\mathfrak{r}_{2k}}|^2 + 2\mu_2\epsilon^{9/4}\psi_{\mathfrak{r}_{2k}}^{-(1+\epsilon^2)}\mathrm{Re}^2\big[D_z\psi_{\mathfrak{r}_{2k}}\eta_{\mathfrak{r}_{2k}}\big]\\&\quad+a\psi_{\mathfrak{r}_{2k+1}}^{-1}\mathrm{Re}^2\big[D_z\psi_{\mathfrak{r}_{2k+1}}\eta_{\mathfrak{r}_{2k+1}}\big] - \Big[\Big(\frac{\epsilon}{2\epsilon_4}\Big)^{\epsilon^2}-1\Big]|\eta_{\mathfrak{r}_{2k+1}}|^2\\&\quad+(1+a^{-1})O\big(|\zeta_{\mathfrak{r}_{2k+1}}|^4\big).\end{aligned}$$

Choosing a small enough in terms of ϵ and ϵ_4, we deduce the analog of (9.33), i.e.

$$\lim_{t\to\mathfrak{r}_{2k+1}+}\bar{\Delta}_t \le \bar{\Delta}_{\mathfrak{r}_{2k+1}} + C|\zeta_{\mathfrak{r}_{2k+1}}|^4. \tag{9.34}$$

From (9.29) and (9.34), we deduce that, at least, for any $n \geq 0$,

$$\lim_{t \to \tau_n +} \bar{\Delta}_t \leq \bar{\Delta}_{\tau_n} + C\bar{\Gamma}^2_{\tau_n},$$

the constant C here depending on $(\mathbf{A})$, ϵ and ϵ_4, that is

$$\lim_{t \to \tau_n +} \big(\bar{\Delta}_t + \bar{\Gamma}^2_t\big) \leq \bar{\Delta}_{\tau_n} + \bar{\Gamma}^2_{\tau_n} + C\bar{\Gamma}^2_{\tau_n}. \tag{9.35}$$

(Eq. (9.35) must be seen as a version of (7.22).)

Inequality (9.35) is not very helpful. To get rid of the term $C\bar{\Gamma}^2_{\tau_n}$, we shall add a correction to the term $(\bar{\Delta}_t + \bar{\Gamma}^2_t)_{t \geq 0}$.

Choose indeed a non-negative smooth function θ with compact support included in $(0, +\infty)$ such that $\theta(\epsilon_4) = 1$ and $\theta(\epsilon/2) = 3$ and consider the processes

$$\begin{aligned}
\bar{\Phi}^{(1)}_t &= \bar{\Delta}^{(1)}_t + (1 + \theta(\psi_t)C)\big(\bar{\Gamma}^{(1)}_t\big)^2,\\
\bar{\Phi}^{(2)}_t &= \bar{\Delta}^{(2)}_t + (1 + 2C)\big(\bar{\Gamma}^{(2)}_t\big)^2,\\
\bar{\Phi}^{(3)}_t &= \bar{\Delta}^{(3)}_t + (1 + 2C)\big(\bar{\Gamma}^{(3)}_t\big)^2, \quad t \geq 0,
\end{aligned}$$

and define the global process $(\bar{\Phi}_t)_{t \geq 0}$ by gathering the three processes above according to the position of $(\psi_t)_{t \geq 0}$ as done to define $(\bar{\Gamma}_t)_{t \geq 0}$ and $(\bar{\Delta}_t)_{t \geq 0}$.

It is well seen that (9.30) still holds for Φ, i.e.

$$d\left[\exp\left(\int_0^t \alpha L\psi_r dr\right)\big(1 + \bar{\Phi}_t\big)^{1/2}\right] \leq C\bar{\Gamma}_t \exp\left(\int_0^t \alpha L\psi_r dr\right)dt. \tag{9.36}$$

It thus remains to check the boundary conditions. When t tends to $\mathfrak{r}_{2k}+$, $\bar{\Phi}_t$ is given by $\bar{\Phi}^{(2)}_t$ and $\psi_t \to \epsilon/2$. Therefore, by (9.35)

$$\lim_{t \to \mathfrak{r}_{2k}+} \bar{\Phi}_t = \lim_{t \to \mathfrak{r}_{2k}+} \bar{\Phi}^{(2)}_t \leq \bar{\Delta}_{\mathfrak{r}_{2k}} + (1 + 3C)\bar{\Gamma}^2_{\mathfrak{r}_{2k}} = \bar{\Phi}^{(1)}_{\mathfrak{r}_{2k+1}} = \bar{\Phi}_{\mathfrak{r}_{2k}}.$$

Similarly, when t tends to $\mathfrak{r}_{2k+1}+$, $\bar{\Phi}_t$ is given by $\bar{\Phi}^{(1)}_t$ and $\psi_t \to \epsilon_4$. Therefore, by (9.35)

$$\lim_{t \to \mathfrak{r}_{2k+1}+} \bar{\Phi}_t = \lim_{t \to \mathfrak{r}_{2k+1}+} \bar{\Phi}^{(1)}_t \leq \bar{\Delta}_{\mathfrak{r}_{2k+1}} + (1 + 2C)\bar{\Gamma}^2_{\mathfrak{r}_{2k+1}} = \bar{\Phi}^{(2)}_{\mathfrak{r}_{2k+1}} = \bar{\Phi}_{\mathfrak{r}_{2k+1}}.$$

This completes the proof. □

9.6. Proof of the Differentiability Properties.

Proposition 9.6. *Choose $0 < \check{\epsilon} < \epsilon_4 < \epsilon < \min(\epsilon_0, \epsilon'_1)$, with ϵ_0 as in Proposition 8.12 and ϵ'_1 as in Proposition 9.3, and consider a cut-off function φ_1 from $\mathbb{C}^d$ into $[0,1]$ matching 1 on the subset $\{z \in \mathcal{D} : \psi(z) \geq \check{\epsilon}\}$ and vanishing on the subset $\{z \in \mathcal{D} : \psi(z) \leq \check{\epsilon}/2\}$. Consider another cut-off function φ_2 from $\mathbb{C}$ to $\mathbb{C}$, matching 1 on $\{y \in \mathbb{C} : |y| \leq r_0\}$, $r_0 = \sup_{z \in \mathcal{D}} \psi^{1/2}(z)$, and vanishing outside $\{y \in \mathbb{C} : |y| \leq 2r_0\}$.*

For any $k \geq 0$, define on $[\mathfrak{r}_{2k}, \mathfrak{r}_{2k+1}]$, $\check{Z}^\varepsilon$ as the solution of

$$
\begin{aligned}
& d\check{Z}_t^{s+\varepsilon} \\
& = T(Z_t^s, \check{Z}_t^{s+\varepsilon} - Z_t^s)(\varphi_1 \psi^{1/2})(\check{Z}_t^{s+\varepsilon}) \exp\bigl(P(Z_t^s, \check{Z}_t^{s+\varepsilon} - Z_t^s)\bigr) \\
& \qquad \times \sigma_t \bigl(dB_t + G(Z_t^s, \check{Z}_t^{s+\varepsilon} - Z_t^s)dt\bigr) \\
& \quad + |T|^2(Z_t^s, \check{Z}_t^{s+\varepsilon} - Z_t^s) \exp\bigl(P(Z_t^s, \check{Z}_t^{s+\varepsilon} - Z_t^s)\bigr) \\
& \qquad \times a_t \exp\bigl(-P(Z_t^s, \check{Z}_t^{s+\varepsilon} - Z_t^s)\bigr)\bigl(\varphi_1 D_{\bar{z}}^* \psi\bigr)(\check{Z}_t^{s+\varepsilon})dt, \quad \mathfrak{r}_{2k} \leq t \leq \mathfrak{r}_{2k+1},
\end{aligned}
\tag{9.37}
$$

with $\check{Z}_0^{s+\varepsilon} = \gamma(s+\varepsilon)$ as initial condition. Above, $(P(Z_t^s, \check{Z}_t^{s+\varepsilon} - Z_t^s))_{t\geq 0}$, $(T(Z_t^s, \check{Z}_t^{s+\varepsilon} - Z_t^s))_{t\geq 0}$, and $(G(Z_t^s, \check{Z}_t^{s+\varepsilon} - Z_t^s))_{t\geq 0}$, stand for the different possible perturbations used in Proposition 9.4. Precisely, $(P(Z_t^s, \check{Z}_t^{s+\varepsilon} - Z_t^s))_{t\geq 0}$ is set equal to 0 outside the intervals on which the perturbation of Proposition 9.3 applies, $(T(Z_t^s, \check{Z}_t^{s+\varepsilon} - Z_t^s))_{t\geq 0}$ is set equal to 1 outside the intervals on which the perturbation of Proposition 8.4 applies and $(G(Z_t^s, \check{Z}_t^{s+\varepsilon} - Z_t^s))_{t\geq 0}$ is set equal to 0 outside the intervals on which the perturbation of Propostion 8.7 applies.

On $[\mathfrak{r}_{2k+1}, \mathfrak{r}_{2k+2}]$, define $\check{Z}^{s+\varepsilon}$ as the first coordinate of the pair $(\check{Z}_t^{s+\varepsilon}, \check{Y}_t^\varepsilon)$ solution of

$$
\begin{aligned}
d\check{Z}_t^{s+\varepsilon} &= \sum_{i=1}^2 \varphi_2\bigl[(\check{Y}_t^{s+\varepsilon})^i\bigr] dB_t^i \\
& \quad + \exp\bigl(P(Z_t^s, \check{Z}_t^{s+\varepsilon} - Z_t^s)\bigr) a_t \exp\bigl(-P(Z_t^s, \check{Z}_t^{s+\varepsilon} - Z_t^s)\bigr) D_{\bar{z}}^* \psi(\check{Z}_t^{s+\varepsilon}) dt \\
d\bigl(\check{Y}_t^{s+\varepsilon}\bigr)^i &= D_{\bar{z}} \psi(\check{Z}_t^{s+\varepsilon}) \exp\bigl(\bar{P}(Z_t^s, \check{Z}_t^{s+\varepsilon} - Z_t^s)\bigr) \bar{\sigma}_t d\bar{B}_t^i \\
& \quad + \frac{1}{2} \varphi_2\bigl[\bigl(\check{Y}_t^{s+\varepsilon}\bigr)^i\bigr] \mathrm{Trace}\bigl[\exp\bigl(P(Z_t^s, \check{Z}_t^{s+\varepsilon} - Z_t^s)\bigr) \\
& \qquad \times a_t \exp\bigl(-P(Z_t^s, \check{Z}_t^{s+\varepsilon} - Z_t^s)\bigr) D_{z,\bar{z}}^2 \psi(\check{Z}_t^{s+\varepsilon})\bigr] dt, \\
& \qquad\qquad\qquad\qquad \mathfrak{r}_{2k+1} \leq t \leq \mathfrak{r}_{2k+2}, \quad i = 1, 2,
\end{aligned}
\tag{9.38}
$$

with $\check{Y}_t^{s+\varepsilon} = ((\varphi_1 \psi^{1/2})(\check{Z}_t^{s+\varepsilon}), 0)$ as initial condition. (Above, ψ is understood as any smooth extension with compact support of the original ψ to the whole space $\mathbb{C}^d$. The perturbation $(P(Z_t^s, \check{Z}_t^{s+\varepsilon} - Z_t^s))_{t\geq 0}$ is the same as in (9.37).)

Then, the process $(\check{Z}_t^{s+\varepsilon})_{t\geq 0}$ is twice differentiable in the mean w.r.t. ε, with time continuous first and second order derivatives, and, the process $(\sum_{k\geq 0} \check{Y}_t^{s+\varepsilon} \mathbf{1}_{[\mathfrak{r}_{2k+1}, \mathfrak{r}_{2k+2}]}(t))_{t\geq 0}$ is also twice differentiable w.r.t. ε, with time continuous first and second order derivatives on every $[\mathfrak{r}_{2k+1}, \mathfrak{r}_{2k+2}]$, $k \geq 0$.

Moreover, for any $S > 0$ and any integer $p \geq 1$,

$$
\sup_{0<|\varepsilon'|<|\varepsilon|} \sup_\sigma \mathbb{E}\Bigl[\sup_{0\leq t\leq S} \bigl(|\check{\zeta}_t^{s+\varepsilon'}|^p + |\check{\eta}_t^{s+\varepsilon'}|^p\bigr)\Bigr] < +\infty,
\tag{9.39}
$$

and

$$
\sup_{0<|\varepsilon'|<|\varepsilon|} \sup_\sigma \mathbb{E}\Bigl[\sup_{k\geq 0} \sup_{\mathfrak{r}_{2k+1}\leq t\leq \mathfrak{r}_{2k+2}, t\leq S} \bigl(|\check{\varrho}_t^{s+\varepsilon'}|^p + |\check{\pi}_t^{s+\varepsilon'}|^p\bigr)\Bigr] < +\infty,
\tag{9.40}
$$

and

$$
\lim_{\varepsilon\to 0} \sup_\sigma \mathbb{E}\Bigl[\sup_{0\leq t\leq S} \bigl(\bigl|\check{\zeta}_t^{s+\varepsilon} - \check{\zeta}_t^s\bigr|^p + \bigl|\check{\eta}_t^{s+\varepsilon} - \check{\eta}_t^s|^p\bigr)\Bigr] = 0,
\tag{9.41}
$$

where $\check\zeta_t^{s+\varepsilon} = [d/d\varepsilon][\check Z_t^{s+\varepsilon}]$, $\check\varrho_t^{s+\varepsilon} = [d/d\varepsilon][\check Y_t^{s+\varepsilon}]\mathbf{1}_{[\mathfrak{r}_{2k+1},\mathfrak{r}_{2k+2}]}(t)$, *and* $\check\eta_t^{s+\varepsilon} = [d^2/d\varepsilon^2][\check Z_t^{s+\varepsilon}]$, $\check\pi_t^{s+\varepsilon} = [d^2/d\varepsilon^2][\check Y_t^{s+\varepsilon}]\mathbf{1}_{[\mathfrak{r}_{2k+1},\mathfrak{r}_{2k+2}]}(t)$, $t \geq 0$, $k \geq 0$.

Proof. We first establish differentiability in probability. By Theorem 7.4, twice differentiability in probability holds on $[0,\mathfrak{r}_1]$, i.e. $(\check\zeta_t^{s+\varepsilon})_{0\leq t\leq \mathfrak{r}_1}$ and $(\check\eta_t^{s+\varepsilon})_{0\leq t\leq \mathfrak{r}_1}$ exist for any ε in the neighborhood of 0, and, for any $S>0$,

$$\lim_{\varepsilon'\to 0,\varepsilon'\neq 0} \sup_{0\leq t\leq S\wedge \mathfrak{r}_1} \left\{ \left|\delta_{\varepsilon'}\check Z_t^{s+\varepsilon} - \check\zeta_t^s\right| + \left|\delta_{\varepsilon'}\check\zeta_t^{s+\varepsilon} - \check\eta_t^{s+\varepsilon}\right| \right\} = 0,$$

in $\mathbb{P}$-probability, i.e. in the sense of (7.7).

In particular, in $\mathbb{P}$-probability,

$$\lim_{\varepsilon'\to 0,\varepsilon'\neq 0} \left\{ \left|\delta_{\varepsilon'}\check Z_{S\wedge\mathfrak{r}_1}^{s+\varepsilon} - \check\zeta_{S\wedge\mathfrak{r}_1}^{s+\varepsilon}\right| + \left|\delta_{\varepsilon'}\check\zeta_{S\wedge\mathfrak{r}_1}^{s+\varepsilon} - \check\eta_{S\wedge\mathfrak{r}_1}^{s+\varepsilon}\right| \right\} = 0,$$

so that we can apply Theorem 7.4 again, but on the time interval $[\mathfrak{r}_1,\mathfrak{r}_2]\cap[0,S]$, or equivalently on $[\mathfrak{r}_1,\mathfrak{r}_2\wedge S]$ and on the event $\{\mathfrak{r}_1 \leq S\}$. Indeed, the dynamics of $(\check Z^{s+\varepsilon},\check Y^{s+\varepsilon})$ on $[\mathfrak{r}_1,\mathfrak{r}_2]\cap[0,S]$ are given by (9.38): Eq. (9.38) satisfies Theorem 7.4. We deduce that $(\check\zeta_t^{s+\varepsilon},\check\rho_t^{s+\varepsilon})_{\mathfrak{r}_1\leq t\leq\mathfrak{r}_2,t\leq S}$ and $(\check\eta_t^{s+\varepsilon},\check\pi_t^{s+\varepsilon})_{\mathfrak{r}_1\leq t\leq\mathfrak{r}_2,t\leq S}$ exist and

$$\begin{aligned}\lim_{\varepsilon'\to 0,\varepsilon'\neq 0} \sup_{\mathfrak{r}_1\leq t\leq\mathfrak{r}_2,t\leq S} \Big\{ &\left|(\delta_{\varepsilon'}\check Z_t^{s+\varepsilon},\delta_{\varepsilon'}\check Y_t^{s+\varepsilon}) - (\check\zeta_t^{s+\varepsilon},\check\rho_t^{s+\varepsilon})\right| \\ &+ \left|(\delta_{\varepsilon'}\check\zeta_t^{s+\varepsilon},\delta_{\varepsilon'}\check\rho_t^{s+\varepsilon}) - (\check\eta_t^{s+\varepsilon},\check\pi_t^{s+\varepsilon})\right| \Big\} = 0,\end{aligned}$$

in $\mathbb{P}$-probability. Then, the procedure can be applied again but on $[\mathfrak{r}_2,\mathfrak{r}_3]\cap[0,S]$, and so on by induction. This proves that twice differentiability in probability holds for the pair process $(\check Z_{t\wedge\mathfrak{r}_n}^{s+\varepsilon},\sum_{k\geq 0}\check Y_{t\wedge\mathfrak{r}_n}^{s+\varepsilon}\mathbf{1}_{[\mathfrak{r}_{2k+1},\mathfrak{r}_{2k+2}]}(t\wedge\mathfrak{r}_n))_{0\leq t\leq S}$, $n\geq 0$. Since $\mathfrak{r}_n\to+\infty$ a.s., twice differentiability in probability follows on the whole $[0,S]$, for any $S>0$. (We emphasize that $\mathfrak{r}_n\to+\infty$ a.s. since the process $(\psi(Z_t^s))_{t\geq 0}$ is a.s. continuous: it cannot switch from ϵ_4 to $\epsilon/2$ an infinite number of times on a compact set.) Twice differentiability in the mean will follow from (9.39), (9.40) and (7.10).

To prove (9.39), we emphasize that, for any $k\geq 0$, we can find a constant C, independent of ε, γ, k and σ, such that, on each $[\mathfrak{r}_{2k},\mathfrak{r}_{2k+1})$[26],

$$d\Big[\exp(-Ct)\left|\check\zeta_t^{s+\varepsilon}\right|^{2p}\Big] \leq dm_t, \quad \mathfrak{r}_{2k}\leq t<\mathfrak{r}_{2k+1}, \tag{9.42}$$

$(m_t)_{\mathfrak{r}_{2k}\leq t<\mathfrak{r}_{2k+1}}$ standing for a generic martingale term. (The proof is the same as the proof of Corollary 7.5.)

Similarly, up to a modification of the constant C, on each $[\mathfrak{r}_{2k+1},\mathfrak{r}_{2k+2})$, $k\geq 0$,

$$d\Big[\exp(-Ct)\big(\left|\check\zeta_t^{s+\varepsilon}\right|^{2p} + \left|\check\varrho_t^{s+\varepsilon}\right|^{2p}\big)\Big] \leq dm_t, \quad \mathfrak{r}_{2k+1}\leq t<\mathfrak{r}_{2k+2}. \tag{9.43}$$

To gather (9.42) and (9.43), it is sufficient to check what happens at boundary times $\mathfrak{r}_n$, $n\geq 0$. The relationship $\check Y_{\mathfrak{r}_{2k+1}}^{s+\varepsilon} = ((\varphi_1\psi^{1/2})(\check Z_{\mathfrak{r}_{2k+1}}^{s+\varepsilon}),0)$ yields

$$\left|\check\varrho_{\mathfrak{r}_{2k+1}}^{s+\varepsilon}\right| = \left|\mathrm{Re}\big[D_z\big(\varphi_1\psi^{1/2}\big)(\check Z_{\mathfrak{r}_{2k+1}}^{s+\varepsilon})\check\zeta_{\mathfrak{r}_{2k+1}}^{s+\varepsilon}\big]\right| \leq C'\left|\check\zeta_{\mathfrak{r}_{2k+1}}^{s+\varepsilon}\right|,$$

for some constant C' (independent of ε, γ, k and σ).

[26]Here, we feel simpler to use right-continuous versions of the processes at hand. Actually, this has an interest for $(\check\varrho_t^{s+\varepsilon})_{t\geq 0}$ only since $(\check\zeta_t^{s+\varepsilon})_{t\geq 0}$ is continuous.

Below, we consider a non-negative smooth function θ with values in $[0,1]$, matching 1 in ϵ_4 and 0 in $\epsilon/2$. Then, for any $k \geq 0$,

$$
\begin{aligned}
&\lim_{t \to \mathfrak{r}_{2k+1}-} \Big[\big(1 + C'\theta\big(\psi(\check{Z}_t^s)\big)\big) \big|\check{\zeta}_t^{s+\varepsilon}\big|^{2p} \Big] \geq \big|\check{\zeta}_{\mathfrak{r}_{2k+1}}^{s+\varepsilon}\big|^{2p} + \big|\check{\varrho}_{\mathfrak{r}_{2k+1}}^{s+\varepsilon}\big|^{2p}, \\
&\lim_{t \to \mathfrak{r}_{2k+2}-} \Big[\big|\check{\zeta}_t^{s+\varepsilon}\big|^{2p} + \big|\check{\varrho}_t^{s+\varepsilon}\big|^{2p} \Big] \geq \big|\check{\zeta}_{\mathfrak{r}_{2k+2}}^{s+\varepsilon}\big|^{2p} = \big(1 + C'\theta\big(\psi(\check{Z}_{\mathfrak{r}_{2k+2}}^s)\big)\big) \big|\check{\zeta}_{\mathfrak{r}_{2k+2}}^{s+\varepsilon}\big|^{2p}.
\end{aligned} \tag{9.44}
$$

Indeed, $\psi(\check{Z}_{\mathfrak{r}_{2k+1}}^s) = \epsilon_4$ and $\psi(\check{Z}_{\mathfrak{r}_{2k+2}}^s) = \epsilon/2$, $k \geq 0$. (Obviously, $(\check{Z}_t^{s+\varepsilon})_{t \geq 0}$ is continuous in time.) Now, it remains to see that

$$
d\Big[\exp(-Ct)\big(1 + C'\theta\big(\psi(\check{Z}_t^s)\big)\big)\big|\check{\zeta}_t^{s+\varepsilon}\big|^{2p}\Big] \leq dm_t, \quad \mathfrak{r}_{2k} \leq t < \mathfrak{r}_{2k+1}, \quad k \geq 0,
$$

for a possibly new value of C. (This follows from Itô's formula.)

Set finally

$$
M_t^p := \begin{cases} \exp(-Ct)\big(1 + C'\theta\big(\psi(\check{Z}_t^s)\big)\big)\big|\check{\zeta}_t^{s+\varepsilon}\big|^{2p}, \ \mathfrak{r}_{2k} \leq t < \mathfrak{r}_{2k+1}, \\ \exp(-Ct)\big(\big|\check{\zeta}_t^{s+\varepsilon}\big|^{2p} + \big|\check{\varrho}_t^{s+\varepsilon}\big|^{2p}\big), \ \mathfrak{r}_{2k+1} \leq t < \mathfrak{r}_{2k+2}, \end{cases}, \quad k \geq 0.
$$

Then, for any $n \geq 0$, $t \mapsto \mathbb{E}[M_{t \wedge \mathfrak{r}_n}]$ is non-increasing. (Use the martingale property and (9.44)). This proves the part related to the first-order derivatives in (9.39) and (9.40), but with the supremum outside the expectation. To get the supremum inside the expectation, we can use so-called Doob's inequality. It says that, for any square integrable progressively-measurable process $(H_t)_{0 \leq t \leq S}$ with values in $\mathbb{C}^d$,

$$
\mathbb{E}\bigg[\sup_{0 \leq t \leq S}\bigg|\int_0^t \langle H_s, dB_s\rangle\bigg|^2\bigg] \leq c\mathbb{E}\int_0^S |H_t|^2 dt,
$$

for some universal $c > 0$. We then choose $(m_t)_{0 \leq t \leq S}$ for $(\int_0^t \langle H_s, dB_s\rangle)_{0 \leq t \leq S}$. We notice that the corresponding process $(H_t)_{0 \leq t \leq S}$ is always bounded by $C'|\check{\zeta}^{s+\varepsilon}|^{2p}$ for $t \in [\mathfrak{r}_{2k}, \mathfrak{r}_{2k+1}] \cap [0,S]$, $k \geq 0$, and by $C'(|\check{\zeta}^{s+\varepsilon}|^{2p} + |\check{\rho}^{s+\varepsilon}|^{2p})$ for $t \in [\mathfrak{r}_{2k+1}, \mathfrak{r}_{2k+2}] \cap [0,S]$, $k \geq 0$, for some constant C' independent of ε, γ, k and σ. Using the bounds for $(\mathbb{E}[M_t^{2p}])_{0 \leq t \leq S}$, (9.39) and (9.40) follow. A similar argument holds for the second-order derivatives (handling the boundary condition by considering $(|\check{\zeta}_t^{s+\varepsilon}|^{4p})_{t \geq 0}$ as in the proof of Corollary 9.5).

We finally turn to (9.41). It relies on the stability property of SDEs. (See Proposition 7.1.) Basically, Proposition 7.1 applies on any interval $[\mathfrak{r}_n, \mathfrak{r}_{n+1}]$. By induction, we obtain

$$
\forall n \geq 1, \quad \lim_{\varepsilon \to 0} \mathbb{E}\big[\sup_{0 \leq t \leq S}\big(\big|\check{\zeta}_t^{s+\varepsilon} - \check{\zeta}_t^s\big|^p + \big|\check{\eta}_t^{s+\varepsilon} - \check{\eta}_t^s|^p\big); S \leq \mathfrak{r}_n\big] = 0. \tag{9.45}
$$

To get the same estimate but on the whole space, we first notice that

$$
\lim_{n \to +\infty} \sup_\sigma \mathbb{P}\{S \leq \mathfrak{r}_n\} = 1. \tag{9.46}
$$

Eq. (9.46) follows from a tightness argument. Since the coefficients of $(Z_t^s)_{t \geq 0}$ are bounded, uniformly in σ, the paths of $(Z_t^s)_{0 \leq t \leq S}$ are continuous, uniformly in σ, with large probability: specifically, given a small positive real ν, we can find a compact subset $\mathcal{K} \subset \mathcal{C}([0,S], \mathbb{C}^d)$, such that, for any σ, $(Z_t^s)_{0 \leq t \leq S}$ belongs to $\mathcal{K}$ with probability greater than $1 - \nu$. To prove (9.46), it then remains to see that $\mathfrak{r}_{2n}/n$ is greater than the smallest amount of time $(Z_t^s)_{t \geq 0}$ needs to switch from ϵ_4 to $\epsilon/2$: clearly, on $[0,S]$, this smallest amount of time is controlled from below in terms of the modulus of continuity of $(Z_t^s)_{0 \leq t \leq S}$ only. In particular, when

$(Z_t^s)_{0\le t\le S}$ belongs to $\mathcal{K}$, S must be less than $\mathfrak{r}_{2n}$ for n larger than some n_0, n_0 depending on $\mathcal{K}$ ans S only.

In particular,

$$\lim_{n\to+\infty}\sup_{\sigma}\mathbb{P}\{S>\mathfrak{r}_n\}=0.$$

By (9.39), (9.40) and Cauchy-Schwarz inequality,

$$\lim_{n\to+\infty}\sup_{\sigma}\mathbb{E}\Big[\sup_{0\le t\le S}\big(|\check{\varrho}_t^{s+\varepsilon'}|^p+|\check{\pi}_t^{s+\varepsilon'}|^p\big);S>\mathfrak{r}_n\Big]=0, \tag{9.47}$$

uniformly in ε' in a neighborhood of 0.

By (9.45) and (9.47), we complete the proof of (9.41).

□

We are now in position to justfity the *meta-statements*:

Corollary 9.7. *Keep the assumption and notation of Propositions 9.4 and 9.6. Then, for any $S>0$ and for $\check{\varepsilon}$ as in Proposition 9.6, there exist a descreasing sequence of positive reals $(\varepsilon_n)_{n\ge1}$, a countable family of increasing events $(\Omega_n)_{n\ge1}$ (i.e. $\Omega_n\subset\Omega_{n+1}$, $n\ge1$), such that $\mathbb{P}(\Omega_n)\to1$ as $n\to+\infty$, and continuous processes $((\zeta_t^{s+\varepsilon})_{0\le t\le S},((\rho_t^{s+\varepsilon})_{\mathfrak{r}_{2k+1}\le t\le\mathfrak{r}_{2k+2},t\le S})_{k\ge0})_{|\varepsilon|<\varepsilon_0}$ and $((\eta_t^{s+\varepsilon})_{0\le t\le S},((\pi_t^{s+\varepsilon})_{\mathfrak{r}_{2k+1}\le t\le\mathfrak{r}_{2k+2},t\le S})_{k\ge0})_{|\varepsilon|<\varepsilon_0}$ such that, for any $n\ge1$, $((Z_t^{s+\varepsilon})_{0\le t\le S})_{|\varepsilon|<\varepsilon_n}$ is twice differentiable in probability on the event Ω_n, with $((\zeta_t^{s+\varepsilon})_{0\le t\le S})_{|\varepsilon|<\varepsilon_n}$ and $((\eta_t^{s+\varepsilon})_{0\le t\le S})_{|\varepsilon|<\varepsilon_n}$ as first and second order derivatives, that is, with the notations of Theorem 7.4,*

$$\forall\varepsilon\in(-\varepsilon_n,\varepsilon_n),\ \forall\nu>0,\ \lim_{\varepsilon'\to0,\varepsilon'\ne0}\mathbb{P}\Big\{\sup_{0\le t\le S}\big|\delta_{\varepsilon'}Z_t^{s+\varepsilon}-\zeta_t^{s+\varepsilon}\big|>\nu,\Omega_n\Big\}=0,$$
$$\lim_{\varepsilon'\to0,\varepsilon'\ne0}\mathbb{P}\Big\{\sup_{0\le t\le S}\big|\delta_{\varepsilon'}\zeta_t^{s+\varepsilon}-\eta_t^{s+\varepsilon}\big|>\nu,\Omega_n\Big\}=0,$$

and, for every $k\ge0$ and $n\ge1$, the family $((Y_t^{s+\varepsilon})_{\mathfrak{r}_{2k}\le t\le\mathfrak{r}_{2k+1},t\le S})_{|\varepsilon|<\varepsilon_n}$ is twice differentiable in probability on Ω_n, with $((\rho_t^{s+\varepsilon})_{\mathfrak{r}_{2k}\le t\le\mathfrak{r}_{2k+1},t\le S})_{|\varepsilon|<\varepsilon_n}$ and $((\pi_t^{s+\varepsilon})_{\mathfrak{r}_{2k}\le t\le\mathfrak{r}_{2k+1},t\le S})_{|\varepsilon|<\varepsilon_n}$ as first and second order derivatives.

Moreover, on each Ω_n, the dynamics of the processes $((\zeta_t^{s+\varepsilon})_{0\le t\le S})_{|\varepsilon|<\varepsilon_n}$ and $((\eta_t^{s+\varepsilon})_{0\le t\le S})_{|\varepsilon|<\varepsilon_n}$ are obtained by differentiating w.r.t. ε the dynamics of $((Z_t^{s+\varepsilon})_{0\le t\le S})_{|\varepsilon|<\varepsilon_n}$ formally, as done in the meta-part *of Section 8. The same holds for the processes $((\rho_t^{s+\varepsilon})_{\mathfrak{r}_{2k+1}\le t\le\mathfrak{r}_{2k+2},t\le S})_{k\ge0})_{|\varepsilon|<\varepsilon_n}$ and $((\pi_t^{s+\varepsilon})_{\mathfrak{r}_{2k+1}\le t\le\mathfrak{r}_{2k+2},t\le S})_{k\ge0})_{|\varepsilon|<\varepsilon_n}$.*

Finally, a.s.,

$$\begin{aligned}
\zeta_t^s&=\frac{d}{d\varepsilon}\big[\check{Z}_t^{s+\varepsilon}\big]_{|\varepsilon=0},\quad \eta_t^s=\frac{d^2}{d\varepsilon^2}\big[\check{Z}_t^{s+\varepsilon}\big],\quad t\ge0,\\
\rho_t^s&=\frac{d}{d\varepsilon}\big[\check{Y}_t^{s+\varepsilon}\big]_{|\varepsilon=0},\quad \pi_t^s=\frac{d^2}{d\varepsilon^2}\big[\check{Y}_t^{s+\varepsilon}\big],\quad \mathfrak{r}_{2k+1}\le t\le\mathfrak{r}_{2k+2},\quad k\ge0.
\end{aligned} \tag{9.48}$$

Before we make the proof, we emphasize the following: the reader may worry about the properties of differentiability of the processes $(Z_t^{s+\varepsilon})_{t\ge0}$ and $((Y_t^{s+\varepsilon})_{\mathfrak{r}_{2k}\le t\le\mathfrak{r}_{2k+1}})_{k\ge0}$ at $\varepsilon=0$. Indeed, we here discussed the notion of differentiability in probability only whereas we used the notion of differentiability in the mean in the meta-statements of Section 8. The reason is the following: all the differentiations we perform below under the symbol $\mathbb{E}$ hold on the families $(\check{Z}_t^{s+\varepsilon})_{t\ge0}$ and $((\check{Y}_t^{s+\varepsilon})_{\mathfrak{r}_{2k}\le t\le\mathfrak{r}_{2k+1}})_{k\ge0}$ only, so that differentiability in the mean of

$(Z_t^{s+\varepsilon})_{t\geq 0}$ and $((Y_t^{s+\varepsilon})_{\mathfrak{r}_{2k}\leq t\leq \mathfrak{r}_{2k+1}})_{k\geq 0}$ is useless. By Proposition 9.6, the families $(\check{Z}_t^{s+\varepsilon})_{t\geq 0}$ and $((\check{Y}_t^{s+\varepsilon})_{\mathfrak{r}_{2k}\leq t\leq \mathfrak{r}_{2k+1}})_{k\geq 0}$ are known to be differentiable in the mean.

Proof. For an arbitrary $\check{\epsilon}$ as in the statement of Proposition 9.6 we know that $(Z_t^s)_{t\geq 0}$ and $(\check{Z}_t^s)_{t\geq 0}$ coincide. (Cut-off functions match 1 because of the stopping times.) Similarly, $((Y_t^s)_{\mathfrak{r}_{2k+1}\leq t\leq \mathfrak{r}_{2k+2}})_{k\geq 0}$ and $((\check{Y}_t^s)_{\mathfrak{r}_{2k+1}\leq t\leq \mathfrak{r}_{2k+2}})_{k\geq 0}$ coincide.

By Theorem 7.2, we know that the mappings $((t,\varepsilon)\in \mathbb{R}_+\times[-\varepsilon_0,\varepsilon_0]\mapsto \check{Z}_t^{s+\varepsilon}$ are once-continuously differentiable for every $\check{\epsilon}$ as in Proposition 9.6. (Here ε_0 stands for a small enough positive real such that $[s-\varepsilon,s+\varepsilon]\subset[-1,1]$). In particular, they are continuous, so that $\sup_{|\varepsilon'|<\varepsilon}\sup_{0\leq t\leq S}|\check{Z}_t^{s+\varepsilon'}-\check{Z}_t^s|$ tends to 0 a.s. as ε tends to 0. Therefore, we can find ε_n small enough such that the event

$$\mathcal{N}_n := \big\{\inf_{|\varepsilon'|<\varepsilon_n}\inf_{k\geq 0}\inf_{\mathfrak{r}_{2k}\leq t\leq \mathfrak{r}_{2k+1},t\leq S}\psi(\check{Z}_t^{s+\varepsilon'})\leq \check{\epsilon}\big\},$$

has probability less than $1/n$.

Set $\Omega_n=(\mathcal{N}_n)^{\complement}$ so that $\mathbb{P}(\Omega_n)\geq 1-1/n$. On Ω_n, $(\check{Z}_t^{s+\varepsilon})_{0\leq t\leq S}$ coincide with $(Z_t^{s+\varepsilon})_{0\leq t\leq S}$ and $((Y_t^{s+\varepsilon})_{\mathfrak{r}_{2k+1}\leq t\leq \mathfrak{r}_{2k+2},t\leq S})_{k\geq 0}$ coincide with the process $((\check{Y}_t^{s+\varepsilon})_{\mathfrak{r}_{2k+1}\leq t\leq \mathfrak{r}_{2k+2},t\leq S})_{k\geq 0}$ for any $\varepsilon\in(-\varepsilon_n,\varepsilon_n)$. (Indeed, on each $[\mathfrak{r}_{2k},\mathfrak{r}_{2k+1}]\cap[0,S]$, $k\geq 0$, the process $(\psi(Z_t^{s+\varepsilon}))_{\mathfrak{r}_{2k}\leq t\leq \mathfrak{r}_{2k+1},t\leq S}$ is above $\check{\epsilon}$ so that $\varphi_1(\check{Z}_t^{s+\varepsilon})$ in (9.37) and in the initial condition of (9.38) matches 1. As a consequence, on each $[\mathfrak{r}_{2k+1},\mathfrak{r}_{2k+2}]\cap[0,S]$, $k\geq 0$, $|\check{Y}_t^{s+\varepsilon}|^2=\psi(\check{Z}_t^{s+\varepsilon})$.) Twice differentiability in probability of $(Z_t^{s+\varepsilon})_{0\leq t\leq S}$ on Ω_n easily follows.

We now check that, on each Ω_n, $n\geq 1$, the dynamics of the derivatives of $(\check{Z}_t^{s+\varepsilon})_{0\leq t\leq S}$ w.r.t. $\varepsilon\in(-\varepsilon_n,\varepsilon_n)$ are obtained by differentiating the dynamics of $(Z_t^{s+\varepsilon})_{0\leq t\leq S}$ formally. This is well-seen since the dynamics of the derivatives of $(\check{Z}_t^{s+\varepsilon})_{0\leq t\leq S}$ are obtained by differentiating the dynamics of $(Z_t^{s+\varepsilon})_{0\leq t\leq S}$ formally and since the cut-off functions φ_1 and φ_2 in the dynamics of $(\check{Z}_t^{s+\varepsilon})_{0\leq t\leq S}$ match 1 on Ω_n.

In particular, on each Ω_n, $n\geq 1$, the derivatives of $(Z_t^{s+\varepsilon})_{0\leq t\leq S}$ at $\varepsilon=0$ and the derivatives of $(\check{Z}_t^{s+\varepsilon})_{0\leq t\leq S}$ at $\varepsilon=0$ coincide. Taking the union over $n\geq 1$, this shows that equality holds almost-surely.

A similar argument holds for $((Y_t^{s+\varepsilon})_{\mathfrak{r}_{2k+1}\leq t\leq \mathfrak{r}_{2k+2},t\leq S})_{k\geq 0}$.

□

9.7. **Differentiability under the symbol $\mathbb{E}$.** We now claim

Proposition 9.8. *With the choice made for $(Z_t^s)_{t\geq 0}$ and $(Z_t^{s+\varepsilon})_{t\geq 0}$ in Proposition 9.4, for a smooth path γ from $[-1,1]$ into $\{z\in\mathcal{D}:\psi(z)>\epsilon_4\}$ and for a given $s\in[-1,1]$, define $\hat{V}_S^\sigma$, V_S^σ and V as in Proposition 8.14. Then, the conclusion of Proposition 8.14 is still true.*

Sketch of the Proof. The proof follows the argument used to establish Proposition 9.1. (See (9.4), (9.5), (9.6) and (9.7).)

Consider $(Z^s_t)_{t\geq 0}$ and define the process

$$W_t = \sum_{n\geq 0}\Big(\int_0^t \mathbf{1}_{\{\mathfrak{r}_{2n}\leq r<\mathfrak{r}_{2n+1}\}} dB_r\Big) \\ + \sum_{i=1,2}\sum_{n\geq 0}\Big(\int_0^t \mathbf{1}_{\{\mathfrak{r}_{2n+1}\leq r<\mathfrak{r}_{2n+2}\}}\big(\frac{Y^i_r}{|Y_r|}\mathbf{1}_{\{|Y_r|>0\}} + \frac{1}{\sqrt{2}}\mathbf{1}_{\{|Y_r|=0\}}\big)dB^i_r\Big), \\ t\geq 0.$$

Then, $(W_t)_{t\geq 0}$ is a complex Brownian motion of dimension d. Moreover,

$$dZ^s_t = \psi^{1/2}(Z^s_t)dW_t + a_t D^*_{\bar{z}}\psi(Z^s_t)dt, \quad t\geq 0.$$

Therefore, for $(Z^s_t)_{t\geq 0}$, everything works as in Proposition 8.14 but with $(B_t)_{t\geq 0}$ replaced by $(W_t)_{t\geq 0}$.

A similar argument holds for $(Z^{s+\varepsilon}_t)_{t\geq 0}$ w.r.t. some $(W^\varepsilon_t)_{t\geq 0}$ (obtained in a similar way). To do so, we emphasize that $(\langle \bar{G}(Z^s_t, Z^{s+\varepsilon}_t - Z^s_t), dB_t\rangle)_{t\geq 0}$ in (8.77) is equal to $(\langle \bar{G}(Z^s_t, Z^{s+\varepsilon}_t - Z^s_t), dW^\varepsilon_t\rangle)_{t\geq 0}$ since G is set equal to 0 on $[\mathfrak{r}_{2n+1}, \mathfrak{r}_{2n+2}]$, $n\geq 0$.

□

We now deduce

Proposition 9.9. *Keep the assumption and notation of Proposition 9.8 and consider in particular a smooth path γ from $[-1,1]$ into $\{z\in\mathcal{D} : \psi(z) > \epsilon_4\}$. Then, there exists a constant $C>0$, depending on (**A**) only, such that, for any $S>0$, the function $s\in(-1,1)\mapsto V_S(\gamma(s)) + C\int_0^s|\gamma'(r)|dr$ is non-decreasing, the function $s\in(-1,1)\mapsto V_S(\gamma(s)) - C\int_0^s|\gamma'(r)|dr$ is non-increasing and the function $s\in(-1,1)\mapsto V_S(\gamma(s)) + C\int_0^s[(s-r)(|\gamma'(r)|^2+|\gamma''(r)|)]dr$ is convex.*

Proof. It is sufficient to find some constant C, depending on (**A**) only, such that for any $s\in(-1,1)$,

$$\lim_{\varepsilon\to 0}\frac{V_S(\gamma(s+\varepsilon)) - V_S(\gamma(s))}{|\varepsilon|} \geq -C|\gamma'(s)|, \\ \lim_{\varepsilon\to 0}\frac{V_S(\gamma(s+\varepsilon)) + V_S(\gamma(s-\varepsilon)) - 2V_S(\gamma(s))}{\varepsilon^2} \geq -C\big(|\gamma'(s)|^2 + |\gamma''(s)|\big), \tag{9.49}$$

and to prove that $V_S\circ\gamma$ is continuous. To do so, we first claim:

Lemma 9.10. *Choose $\epsilon = \min(\epsilon_0, \epsilon'_1)/2$, with ϵ_0 as in Proposition 8.12 and ϵ'_1 as in Proposition 9.3.*

Define

$$\check{p}^\varepsilon_t = P(Z^s_r, \check{Z}^{s+\varepsilon}_r - Z^s_r),\ \check{\tau}^\varepsilon_t = T(Z^s_r, \check{Z}^{s+\varepsilon}_r - Z^s_r), \\ \check{\Xi}^\varepsilon_t = G(Z^s_r, \check{Z}^{s+\varepsilon}_r - Z^s_r), \quad t\geq 0.$$

For a given smooth cut-off function ρ with values in $[0,1]$ matching the identity on $[1/2,3/2]$ and vanishing outside a compact subset, set as well

$$
\begin{aligned}
&\check{V}_S^{\sigma}(s+\varepsilon) \\
&= \mathbb{E}\int_0^{+\infty}\Bigl[\rho\Bigl(\exp\Bigl(-\int_0^t 2\mathrm{Re}\bigl[\langle\bar{\check{\Xi}}_r^{\varepsilon}, dB_r\rangle\bigr] - \int_0^t |\check{\Xi}_r^{\varepsilon}|^2 dr\Bigr)\Bigr) \\
&\qquad\times \exp\Bigl(\int_0^t |\check{\tau}_r^{\varepsilon}|^2 \mathrm{Trace}[\exp(\check{p}_r^{\varepsilon}) a_r \exp(-\check{p}_r^{\varepsilon}) D^2_{z,\bar{z}}\psi(\check{Z}_r^{s+\varepsilon})] dr\Bigr) \\
&\qquad\times F\bigl(\det(a_t), \exp(\check{p}_t^{\varepsilon}) a_t \exp(-\check{p}_t^{\varepsilon}), \check{Z}_t^{s+\varepsilon}\bigr)\phi\bigl(\frac{\check{\mathfrak{T}}_t^{\varepsilon}}{S}\bigr)\Bigr]|\check{\tau}_t^{\varepsilon}|^2 dt,
\end{aligned}
\tag{9.50}
$$

with $[d/dt](\check{\mathfrak{T}}_t^{\varepsilon}) = (\check{\tau}_t^{\varepsilon})^2$, $t \geq 0$.

*Then, $\sup_{\sigma}[\check{V}_S^{\sigma}(s)] = V_S(\gamma(s))$ and, for ε in the neighborhood of 0, $\sup_{\sigma}[\check{V}_S^{\sigma}(s+\varepsilon)] \leq V_S(\gamma(s+\varepsilon)) + C\varepsilon^3$, for a constant C depending on (**A**) and S only.*

Moreover, we can find a constant C such that

$$
\begin{aligned}
&\lim_{\varepsilon\to 0}\sup_{|\varepsilon'|<|\varepsilon|}\sup_{\sigma}\Bigl|\frac{d}{d\varepsilon'}\bigl[\check{V}_S^{\sigma}\bigl(\gamma(s+\varepsilon')\bigr)\bigr]\Bigr| \\
&\leq \mathbb{E}\Bigl[\int_0^{+\infty}\exp\Bigl(\int_0^t \mathrm{Trace}[a_r D^2_{z,\bar{z}}\psi(Z_r^s)]dr\Bigr) \\
&\qquad\times\Bigl(|\bar{\Gamma}_t| + \int_0^t (1+r^{-1/2})|\bar{\Gamma}_r| dr\Bigr)dt\Bigr],
\end{aligned}
\tag{9.51}
$$

and,

$$
\begin{aligned}
&\lim_{\varepsilon\to 0}\sup_{|\varepsilon'|<|\varepsilon|}\sup_{\sigma}\Bigl|\frac{d^2}{d\varepsilon'^2}\bigl[\check{V}_S^{\sigma}\bigl(\gamma(s+\varepsilon')\bigr)\bigr]\Bigr| \\
&\leq \mathbb{E}\Bigl[\int_0^{+\infty}\exp\Bigl(\int_0^t \mathrm{Trace}[a_r D^2_{z,\bar{z}}\psi(Z_r^s)]dr\Bigr) \\
&\qquad\times\Bigl(|\bar{\Gamma}_t|^2 + |\bar{\Delta}_t| + \int_0^t (1+r^{-1/2})\bigl(|\bar{\Gamma}_r|^2 + |\bar{\Delta}_r|\bigr)dr\Bigr)dt\Bigr].
\end{aligned}
\tag{9.52}
$$

Finally, for every compact interval $I \subset (-1,1)$ and for ε small enough, the quantity $\sup_{\sigma}\sup_{|\varepsilon'|<|\varepsilon|}[|(\partial/\partial\varepsilon')[\check{V}_S^{\sigma}(\gamma(s+\varepsilon'))]|]$ is uniformly bounded w.r.t. $s \in I$. (Pay attention that the definition of $\check{V}_S^{\sigma}$ depends on s itself.)

End of the Proof of Proposition 9.9. Before we prove Lemma 9.10, we complete the proof of Proposition 9.9. Clearly, by Lemma 9.10

$$
\begin{aligned}
\lim_{\varepsilon\to 0}\frac{V_S(\gamma(s+\varepsilon)) - V_S(\gamma(s))}{|\varepsilon|} &\geq \lim_{\varepsilon\to 0}\frac{1}{|\varepsilon|}\bigl[\inf_{\sigma}\bigl(\check{V}^{\sigma}(s+\varepsilon) - \check{V}^{\sigma}(s)\bigr)\bigr] \\
&\geq -\lim_{\varepsilon\to 0}\sup_{|\varepsilon'|<|\varepsilon|}\sup_{\sigma}\Bigl|\frac{d}{d\varepsilon'}\bigl[\check{V}^{\sigma}(s+\varepsilon')\bigr]\Bigr|.
\end{aligned}
$$

By Lemma 9.10, we deduce that

$$
\begin{aligned}
&\lim_{\varepsilon\to 0}\frac{V_S(\gamma(s+\varepsilon))-V_S(\gamma(s))}{|\varepsilon|}\\
&\quad\geq -\sup_\sigma \mathbb{E}\Big[\int_0^{+\infty}\exp\Big(\int_0^t \mathrm{Trace}\big[a_r D^2_{z,\bar z}\psi(Z_r^\sigma)\big]dr\Big)\\
&\qquad\qquad\times\Big(|\bar\Gamma_t|+\int_0^t(1+r^{-1/2})|\bar\Gamma_r|dr\Big)dt\Big].
\end{aligned}
$$

By Proposition 9.4, we deduce that there exists a constant C, depending on $(\mathbf{A})$ only, such that the first inequality in (9.49) holds. The same strategy holds to prove the second inequality in (9.49).

It remains to prove that $V_S\circ\gamma$ is continuous. Basically,

$$
\begin{aligned}
V_S(\gamma(s+\varepsilon))-V_S(\gamma(s)) &\geq \sup_\sigma\big[\check V_S^\sigma(s+\varepsilon)\big]-\sup_\sigma\big[\check V_S^\sigma(s)\big]-C|\varepsilon|^3\\
&\geq -|\varepsilon|\sup_{|\varepsilon'|<|\varepsilon|}\sup_\sigma\big[\big|\frac{\partial \check V_S^\sigma}{\partial\varepsilon'}(s+\varepsilon')\big|\big]-C|\varepsilon|^3.
\end{aligned}
$$

Therefore, for any compact interval $I\subset(-1,1)$, for ε small enough, we can find some constant C' such that

$$
V_S(\gamma(s+\varepsilon))-V_S(\gamma(s))\geq -C'|\varepsilon|,
$$

when s and $s+\varepsilon$ are in I. Exchanging the roles of $s+\varepsilon$ and s, this proves that $V_S\circ\gamma$ is continuous. □

We now prove Lemma 9.10.

Proof of Lemma 9.10. The equality $\sup_\sigma[\check V_S^\sigma(s)]=V_S(\gamma(s))$ is easily taken since $\check V_S^\sigma(s)=\hat V_S^\sigma(s)$, with $\hat V_S^\sigma$ as in Proposition 9.8.

We now establish the inequality $\sup_\sigma[\check V_S^\sigma(s+\varepsilon)]\leq V_S(\gamma(s+\varepsilon))+C\varepsilon^3$. It is well-seen that all the terms under the integral symbol in (9.50) are bounded by some constant C depending on $(\mathbf{A})$ and S only.

Therefore, for some $\epsilon'>0$ to be chosen later,

$$
\begin{aligned}
&\check V_S^\sigma(s+\varepsilon)\\
&=\mathbb{E}\Big\{\int_0^{+\infty}\Big[\rho\Big(\exp\Big(-\int_0^t 2\mathrm{Re}\big[\langle\check\Xi_r^\varepsilon,dB_r\rangle\big]-\int_0^t|\check\Xi_r^\varepsilon|^2dr\Big)\Big)\\
&\qquad\times\exp\Big(\int_0^t|\check\tau_r^\varepsilon|^2\mathrm{Trace}[\exp(\check p_r^\varepsilon)a_r\exp(-\check p_r^\varepsilon)D^2_{z,\bar z}\psi(\check Z_r^{s+\varepsilon})]dr\Big)\\
&\qquad\times F\big(\det(a_t),\exp(\check p_t^\varepsilon)a_t\exp(-\check p_t^\varepsilon),\check Z_t^{s+\varepsilon}\big)\phi\big(\frac{\check{\mathfrak{T}}_t^\varepsilon}{S}\big)\Big]|\check\tau_t^\varepsilon|^2dt;\\
&\qquad\qquad\sup_{0\leq t\leq S}|\check Z_t^{s+\varepsilon}-Z_t^s|\leq\epsilon'\Big\}\\
&\quad+O\big(\mathbb{P}\{\sup_{0\leq t\leq S}|\check Z_t^{s+\varepsilon}-Z_t^s|\geq\epsilon'\}\big).
\end{aligned}
\tag{9.53}
$$

(Here, the Landau term $O(\cdots)$ is uniform w.r.t. ε.)

As long as the process $(|\check{Z}^{s+\varepsilon}_t - Z^s_t|)_{t\geq 0}$ stays below ϵ', the process $(|\psi(\check{Z}^{s+\varepsilon}_t) - \psi(Z^s_t)|)_{t\geq 0}$ stays below some $C\epsilon'$, C depending on ψ only. In particular, we can choose ϵ' small enough such that $C\epsilon' < \check{\epsilon}/2$. (See Proposition 9.6 for the definition of $\check{\epsilon}$.)

On each $[\mathfrak{r}_{2k}, \mathfrak{r}_{2k+1}]$, $k \geq 0$, as in Proposition 9.4, the process $(\psi(Z^s_t))_{\mathfrak{r}_{2k}\leq t\leq \mathfrak{r}_{2k+1}}$ is above $\epsilon_4 > 2\check{\epsilon}$. Therefore, on each $[\mathfrak{r}_{2k}, \mathfrak{r}_{2k+1}] \cap [0, S]$, $k \geq 0$, the condition $\sup_{0\leq t\leq S} |\check{Z}^{s+\varepsilon}_t - Z^s_t| \leq \epsilon'$ implies (recall that $a \wedge b$ stands for $\min(a, b)$)

$$\psi(\check{Z}^{s+\varepsilon}_t) > \check{\epsilon}, \quad t \in [\mathfrak{r}_{2k}, \mathfrak{r}_{2k+1}] \cap [0, S],$$

so that $\varphi_1(\check{Z}^{s+\varepsilon}_t)$ in (9.37) and in the initial condition of (9.38) matches 1. As a consequence, on each $[\mathfrak{r}_{2k+1}, \mathfrak{r}_{2k+2}] \cap [0, S]$, $k \geq 0$, the condition $\sup_{0\leq t\leq S} |\check{Z}^{s+\varepsilon}_t - Z^s_t| \leq \epsilon'$ implies

$$|\check{Y}^{s+\varepsilon}_t|^2 = \psi(\check{Z}^{s+\varepsilon}_t), \quad t \in [\mathfrak{r}_{2k+1}, \mathfrak{r}_{2k+2}] \cap [0, S].$$

Finally, under the condition $\sup_{0\leq t\leq S} |\check{Z}^{s+\varepsilon}_t - Z^s_t| \leq \epsilon'$, processes $(\check{Z}^{s+\varepsilon}_t)_{0\leq t\leq S}$ and $(Z^{s+\varepsilon}_t)_{0\leq t\leq S}$ have the same dynamics on the whole $[0, S]$.

As a consequence, the first term in (9.53) is less than $\hat{V}^{\sigma}_S(s + \varepsilon)$. (Use $F \geq 0$ to say so.) It thus remains to bound the second term.

The idea consists in using Markov inequality. For any $p \geq 1$, it says that

$$\mathbb{P}\big\{\sup_{0\leq t\leq S} |\check{Z}^{s+\varepsilon}_t - Z^s_t| \geq \check{\epsilon}/2\big\} \leq 2^p \check{\epsilon}^{-p} \mathbb{E}\big[\sup_{0\leq t\leq S} |\check{Z}^{s+\varepsilon}_t - Z^s_t|^p\big]. \tag{9.54}$$

Using the stability property for SDEs, see Proposition 7.1, we know that

$$\begin{aligned}
&\mathbb{E}\big[\sup_{0\leq t\leq S}\big|\check{Z}^{s+\varepsilon}_t - Z^s_t\big|^p\big] \\
&\leq C\varepsilon^p + C\mathbb{E}\int_0^S \big(\big|\check{Z}^{s+\varepsilon}_r - Z^s_r\big|^p + \big|\check{Y}^{s+\varepsilon}_r - Y^s_r\big|^p\big)dr \\
&\leq C\varepsilon^p\big(1 + \int_0^S \sup_{|\varepsilon'|\leq\varepsilon} \mathbb{E}\big[\big|\check{\zeta}^{s+\varepsilon'}_r\big|^p + \big|\check{\varrho}^{s+\varepsilon'}_r\big|^p\big]dr \leq C\varepsilon^p.
\end{aligned} \tag{9.55}$$

Plugging the above bound in (9.54) and then in (9.53), we complete the proof of the bound $\sup_\sigma[\check{V}^{\sigma}_S(s+\varepsilon)] \leq V_S(\gamma(s+\varepsilon)) + C\varepsilon^3$.

The proof of the inequalities (9.51) is now straightforward: it follows from (8.82), (9.39), (9.41) and (9.48):

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \sup_{|\varepsilon'|<|\varepsilon|} \sup_\sigma \Big|\frac{d}{d\varepsilon}\big[\check{V}^\sigma\big(\gamma(s+\varepsilon)\big)\big]\Big| \\
&\leq \sup_\sigma \mathbb{E}\Big[\int_0^{+\infty} \exp\Big(\int_0^t \mathrm{Trace}[a_r D^2_{z,\bar{z}}\psi(Z^s_r)]dr\Big) \\
&\quad \times \Big(|\zeta^s_t| + \int_0^t |\zeta^s_r|dr + \Big|\int_0^t \mathrm{Re}\big[\langle D_{z'}G(Z^s_r, 0)\zeta^s_r, dB_r\rangle\big]\Big|\Big)dt\Big].
\end{aligned} \tag{9.56}$$

Following the proof of Proposition 8.8 (and specifically using a variant of Lemma 8.5[27]), we obtain

$$\frac{d}{d\varepsilon}\big[V_S\big(\gamma(s+\varepsilon)\big)\big] \leq \mathbb{E}\bigg[\int_0^{+\infty} \exp\bigg(\int_0^t \mathrm{Trace}[a_r D^2_{z,\bar{z}}\psi(Z^s_r)]dr\bigg) \times \bigg(|\bar{\Gamma}_t| + \int_0^t \big(1+r^{-1/2}\big)|\bar{\Gamma}_r|dr\bigg)dt\bigg]. \tag{9.57}$$

The same argument holds for the second-order derivatives.

Finally, for every compact interval $I \subset (-1,1)$ and for ε small enough, the quantity $\sup_\sigma \sup_{|\varepsilon'|<|\varepsilon|}[|(\partial/\partial\varepsilon')[\check{V}^g_S(\gamma(s+\varepsilon'))]|]$ is shown to be uniformly bounded w.r.t. $s \in I$ by a similar argument and by (9.39). □

9.8. **Final Step.** We now complete the proof of Theorem 6.1.

Passing to the limit in $S \to +\infty$ in Proposition 9.9, we deduce that V in Proposition 6.9 satisfies the same property as V_S, i.e. for any smooth curve γ from $[-1,1]$ into $\{z \in \mathcal{D} : \psi(z) > \epsilon_4\}$, the function $s \in [-1,1] \mapsto V(\gamma(s)) + C\int_0^s |\gamma'(r)|dr$ is non-decreasing, the function $s \in [-1,1] \mapsto V(\gamma(s)) - C\int_0^s |\gamma'(r)|dr$ is non-increasing and the function $s \in [-1,1] \mapsto V(\gamma(s)) + C\int_0^s [(s-r)(|\gamma''(r)| + |\gamma'(r)|^2)]dr$ is convex.

Choosing γ as a straight path of the form $s \in [-1,1] \mapsto z + \nu s$, for $\psi(z) > \epsilon_4$ and $\nu \in \mathbb{C}^d$, with $|\nu|$ small enough, we deduce that V is Lipschitz and semi-convex away from the boundary, i.e. on $\{z \in \mathcal{D} : \psi(z) > \epsilon_4\}$. In particular, $v - g + N_0\psi$ in Proposition 6.9 is Lipschitz and semi-convex on $\{z \in \mathcal{D} : \psi(z) > \epsilon_4\}$ as well. By Proposition 6.4 and Remark 6.5, v is $\mathcal{C}^{1,1}$ on $\{z \in \mathcal{D} : \psi(z) > \epsilon_4\}$. Since ϵ_4 may be chosen as small as desired, we deduce that v is $\mathcal{C}^{1,1}$ in $\mathcal{D}$.

We emphasize that the Lipschitz and semi-convexity constants are bounded in terms of (**A**) only on every compact subset. The problem is then to bound the Lipschitz and semi-convexity constants up to the boundary.

To do so, we consider a path γ_0 from $[-1,1]$ into $\{z \in \mathcal{D} : \psi(z) < \epsilon/2\}$, for the same ϵ as in Propositions 9.4 and 9.6. Then, we can define $(Z^s_t)_{0\leq t\leq \mathfrak{r}_1}$ as in (9.23) first, i.e. as the first coordinate of the pair $(Z^s_t, Y^s_t)_{0\leq t\leq\mathfrak{r}_1}$, $\mathfrak{r}_1$ now standing for $\inf\{t \geq 0 : \psi(Z^s_t) > \epsilon/2\}$. and switch to (9.22) from $\mathfrak{r}_1$ to $\mathfrak{r}_2$, with $\mathfrak{r}_2 = \inf\{t \geq \mathfrak{r}_1 : \psi(Z^s_t) < \epsilon_4\}$, and so on... Here, Z^s_0 is chosen as $\gamma_0(s)$ and Y^s_0 is chosen in such a way that $|Y^s_0|^2 = \psi(Z^s_0) = \psi(\gamma(s))$. Obviously, we can apply the same procedure for the perturbed process and first consider $(\check{Z}^{s+\varepsilon}_t, \check{Y}^{s+\varepsilon}_t)_{0\leq t\leq\mathfrak{r}_1}$ as in (9.38).

The whole question then lies in the choice of the initial condition $(\check{Z}^{s+\varepsilon}_0, \check{Y}^{s+\varepsilon}_0)$. Surely, we choose $\check{Z}^{s+\varepsilon}_0$ as $\gamma_0(s+\varepsilon)$ and $\check{Y}^{s+\varepsilon}_0$ such that $|\check{Y}^{s+\varepsilon}_0|^2 = \psi(\check{Z}^{s+\varepsilon}_0)$. Assume therefore that $\check{Y}^{s+\varepsilon}_0 = \gamma_1(s+\varepsilon)$ for some smooth path γ_1 defined on $[-1,1]$ such that $\psi(\gamma_0(s)) = |\gamma_1(s)|^2$, $s \in [-1,1]$. Then, Proposition 9.9 remains true with $\gamma = (\gamma_0, \gamma_1)$, the proof being exactly the same. In particular, the constant C therein depends on (**A**) only (and is independent

[27] In Section 8, the process $(\varsigma_t)_{t\geq 0}$ in the statement of Lemma 8.5 is understood as $(\zeta^s_t)_{t\geq 0}$. Here, ς_t, $t \geq 0$, is to be understood as ζ^s_t or (ζ^s_t, ϱ^s_t) according to the cases: $t \in [\mathfrak{r}_{2k}, \mathfrak{r}_{2k+1}[$ or $t \in [\mathfrak{r}_{2k+1}, \mathfrak{r}_{2k+2}[$, $k \geq 0$. For this reason, it may be simpler to plug $(\bar{\Gamma}_t)_{t\geq 0}$ itself into $(\varsigma_t)_{t\geq 0}$.

However, since it is discontinuous, $(\bar{\Gamma}_t)_{t\geq 0}$ does not satisfy the assumption of Lemma 8.5. Actually, it is sufficient to apply Itô's formula to $((a + m_t + \bar{\Gamma}_t)^{1/2})_{t\geq 0}$ on each $(\mathfrak{r}_{2k}, \mathfrak{r}_{2k+1})$, a standing for a small positive real, and then to check the boundary conditions. In particular, it is useless to localize the proof as done in the proof of Lemma 8.5 since there is no singulariry anymore in the dynamics of the derivative processes.

of the distance of γ_0 to the boundary). Since V is now known to be $\mathcal{C}^{1,1}$ in $\mathcal{D}$ (see Remark 6.5), this may be read as

$$
\begin{aligned}
&\Big|\frac{d[V(\gamma_0(s))]}{ds}\Big| \le C|\gamma'(s)| \quad s\in[-1,1] \\
&\Big|\frac{d^2[V(\gamma_0(s))]}{ds^2}\Big| \le C\big(|\gamma'(s)|^2+|\gamma''(s)|\big) \quad \text{a.e. } s\in[-1,1].
\end{aligned}
\tag{9.58}
$$

To obtain the Lipschitz property up to the boundary, we fix some z with $\psi(z)<\epsilon/2$ and we choose γ as in Proposition 9.1, i.e. $\gamma=(\gamma_0,\gamma_1)$ with $\gamma_0(s)=z+s\nu$, $s\in[-1,1]$, for $\nu\in\mathbb{C}^d$ with a small enough norm, and $\gamma_1=(\gamma_{1,1},0)$, with

$$
(\gamma_{1,1})'(s)=(\bar{\gamma}_{1,1})^{-1}(s)D_z\psi(\gamma_0(s))\nu \quad |\gamma_{1,1}(0)|^2=\psi(z), \quad s\in[-1,1].
$$

Keep in mind that $|\gamma_{1,1}(s)|^2=\psi(\gamma_0(s))$ for $s\in[-1,1]$.

Now, compute for a differentiable function $w(s)$:

$$
\Big|\frac{d[w(s)\psi(\gamma_0(s))]}{ds}\Big| = \Big|\psi(\gamma_0(s))\frac{dw}{ds}(s)+2w(s)\mathrm{Re}\big[D_z\psi(\gamma_0(s))\nu\big]\Big|.
$$

Choose now $w=V\circ\gamma_0$ and deduce from (9.58) that

$$
\begin{aligned}
&\left|\frac{d[V(\gamma_0(s))\psi(\gamma_0(s))]}{ds}\right| \\
&\quad \le C\psi(\gamma_0(s))\big[|\nu|+|\bar{\gamma}_{1,1}^{-1}(s)||D_z\psi(\gamma_0(s))\nu|\big]+C\|V\|_\infty|\nu|.
\end{aligned}
$$

Modifying the constant C if necessary, we deduce that $[\psi V](\gamma_0(s))$ is Lipschitz continuous of constant $C|\nu|$. We emphasize that the constant C is independent of the distance from z to the boundary since $|\psi(\gamma_0(s))\bar{\gamma}_{1,1}^{-1}|=\psi^{1/2}(\gamma_0(s))$ is bounded. This procedure directly applies to Proposition 6.9: we deduce that $v-g+N_0\psi$ is Lipschitz continous up to the boundary. This is the first part in Theorem 8.1.

It now remains to investigate the second-order derivatives. To obtain an estimate that holds up to the boundary, we consider another parameterized curve. Let $(\gamma_0^a,\gamma_{1,1}^a)$ and $(\gamma_0^b,\gamma_{1,1}^b)$ be two pairs with values in $\mathcal{D}\times\mathbb{R}$ such that

$$
\dot{\gamma}_0^i(s)=\gamma_{1,1}^i(s)\nu, \quad \dot{\gamma}_{1,1}^i(s)=\mathrm{Re}\big[D_z\psi(\gamma_0^i(s))\nu\big], \quad i=a,b.
\tag{9.59}
$$

(Pay attention that $\gamma_{1,1}^i$ is real-valued.) The initial boundary condition has the form: $\gamma_0^i(0)=z$ (with $\psi(z)<\epsilon/2$) and $\gamma_{1,1}^i(0)=y_0^i\in\mathbb{R}$, with y_0^i to be chosen later on. Clearly, for each $i=a,b$, the system is (at least) solvable on a small interval around 0. Now,

$$
\begin{aligned}
&\frac{d}{ds}\big[\psi(\gamma_0^i(s))-|\gamma_{1,1}^i(s)|^2\big] \\
&=2\mathrm{Re}\big[D_z\psi(\gamma_0^i(s))\dot{\gamma}_0^i(s)\big]-2\gamma_{1,1}^i(s)\mathrm{Re}\big[D_z\psi(\gamma_0^i(s))\nu\big] \\
&=0.
\end{aligned}
\tag{9.60}
$$

Now, for $w^i = V \circ \gamma_0^i$ and for s in the interval of definition of $(\gamma_0^i, \gamma_{1,1}^i)$,

$$\begin{aligned}
&\frac{d^2}{ds^2}\big[V(\gamma_0^i(s))\big] \\
&= 2\frac{d}{ds}\big\{\gamma_{1,1}^i(s)\mathrm{Re}\big[D_z V(\gamma_0^i(s))\nu\big]\big\} \\
&= 2\mathrm{Re}\big[D_z\psi(\gamma_0^i(s))\nu\big]\mathrm{Re}\big[D_z V(\gamma_0^i(s))\nu\big] + |\gamma_{1,1}^i(s)|^2\big[D^2V(\gamma_0^i(s))\big](\nu),
\end{aligned}$$

where $[D^2V(\gamma_0^i(s))](\nu)$ stands for the action of the second-order derivatives of V at point $\gamma_0^i(s)$ on the vector ν[28]. Choosing $s=0$ and making the sum over $i=a,b$, we obtain:

$$\begin{aligned}
\sum_{i=a,b}\frac{d^2}{ds^2}\big[V(\gamma_0^i(s))\big]_{|s=0} &= 4\mathrm{Re}\big[D_z\psi(z)\nu\big]\mathrm{Re}\big[D_z V(z)\nu\big] \\
&\quad + (|y_0^a|^2 + |y_0^b|^2)\big[D^2V(z)\big](\nu).
\end{aligned}$$

The whole trick now consists in choosing $|y_0^a|^2 = |y_0^b|^2 = \psi(z)/2$ so that

$$\begin{aligned}
&\big[D^2\big(\psi V\big)(z)\big](\nu) \\
&= \big[D^2\psi(z)\big](\nu)V(z) \\
&\quad + 4\mathrm{Re}\big[D_z\psi(z)\nu\big]\mathrm{Re}\big[D_z V(z)\nu\big] + \psi(z)\big[D^2V(z)\big](\nu) \\
&= \big[D^2\psi(z)\big](\nu)V(z) + \sum_{i=a,b}\frac{d^2}{ds^2}\big[V(\gamma_0^i(s))\big]_{|s=0}.
\end{aligned}$$

To apply (9.58), we need to specify what the second coordinate of each γ_1^i is. We set $\gamma_1^i(s) = (\gamma_{1,1}^i(s), (\psi(z)/2)^{1/2})$ for s in the interval of definition of $(\gamma_0^i, \gamma_{1,1}^i)$. By (9.60), it satisfies $\psi(\gamma_0^i(s)) - |\gamma_1^i(s)|^2 = 0$, so that (γ_0^i, γ_1^i), $i=a,b$, is a zero of the function $\Phi(z,y) = \psi(z) - |y|^2$. (In particular, γ_0^i cannot exit from $\mathcal{D}$ and the solution to (9.59) may be extended to the whole $[-1,1]$. Indeed, γ_1^i cannot vanish since $\gamma_{1,2}^i(s) = (\psi(z)/2)^{1/2}$.) We now apply (9.58) (with s in the neighborhood of 0 only). Then, we obtain that $D^2[\psi(z)V(z)](\nu) \geq -C|\nu|^2$, for some constant C, independent from the distance from z to the boundary. Since $\psi V = v - g + N_0\psi$, this proves that the semi-convexity constant of v is uniform up to the boundary. By Proposition 6.4, we complete the proof of Theorem 8.1.

9.9. **Conclusion.** We here paid some price to gather into a single one the two different representations $((Z_t^s)_{\mathfrak{r}_{2k} \leq t < \mathfrak{r}_{2k+1}})_{k\geq 0}$ and $((Z_t^s)_{\mathfrak{r}_{2k+1} \leq t < \mathfrak{r}_{2k+2}})_{k\geq 0}$ according to the position of the process $(Z_t^s)_{t\geq 0}$ inside the domain $\mathcal{D}$.

A natural way to simplify things consists in considering the parameterized representation (9.1) in the whole space and in forgetting the original Eq. (8.1). Actually, this is exactly what Krylov does in the papers mentioned in the references below.

The reason why we here decided to split the representation into two pieces is purely pedagogical even if a bit heavy to detail. Indeed, Section 8 exactly shows what works and fails when dealing with the first approach. In some sense, this may justify in a more understandable way the reason why the parameterized version is the one used by Krylov. We also emphasize that the computations performed in Section 8 for the single process $(Z_t^s)_{t\geq 0}$

[28]That is, $D^2[V(z)](\nu) = \sum_{k,\ell=1}^d (D^2_{z_k,z_\ell}V(z)\nu_k\nu_\ell + D^2_{\bar z_k,z_\ell}V(z)\bar\nu_k\nu_\ell + D^2_{z_k,\bar z_\ell}V(z)\nu_k\bar\nu_\ell + D^2_{\bar z_k,\bar z_\ell}V(z)\bar\nu_k\bar\nu_\ell)$.

turn out to be really cumbersome for the pair process $(Z_t^s, Y_t^s)_{t\geq 0}$: this is another reason why we kept both representations in the whole proof.

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